Filters, Cost Functions, and Controller Structures

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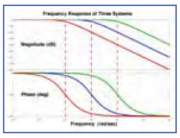
Optimal Control and Estimation MAE 546
Princeton University, 2015

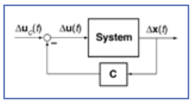
- **Dynamic systems as low-pass filters**
- **Frequency response of dynamic systems**
- **Shaping system response**
  - **LQ regulators with output vector cost functions**
  - **Implicit model-following**
  - **Cost functions with augmented state vector**



$$\min_{\Delta \mathbf{u}} J = \frac{1}{2}\int_0^\infty \left[\Delta \mathbf{x}^T(t)\mathbf{Q}\Delta \mathbf{x}(t) + \Delta \mathbf{u}^T(t)\mathbf{R}\Delta \mathbf{u}(t)\right]dt$$

subject to

$$\Delta \dot{\mathbf{x}}(t) = \mathbf{F}\Delta \mathbf{x}(t) + \mathbf{G}\Delta \mathbf{u}(t)$$

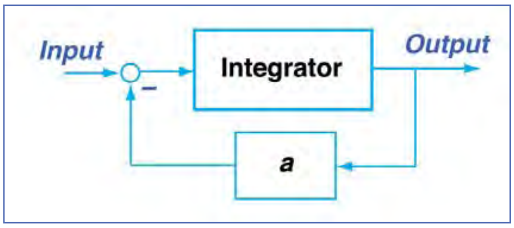


http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

First-Order Low-Pass Filter

Low-Pass Filter

Low-pass filter passes low frequency signals and attenuates high-frequency signals



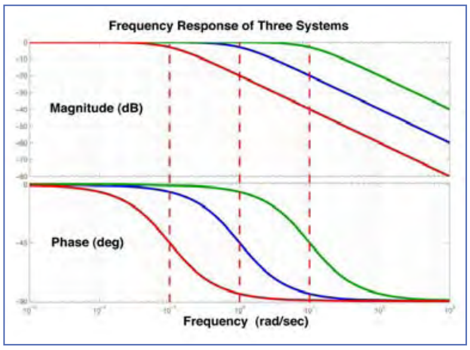
$$\dot{x}(t) = -ax(t) + au(t)$$
$$a = 0.1,\ 1,\ \text{or } 10$$

- **Laplace transform, $x(0) = 0$**

$$x(s) = \frac{a}{(s+a)}u(s)$$

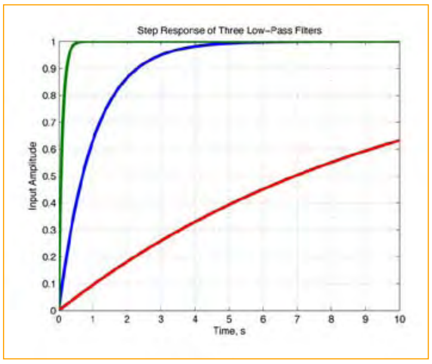
- **Frequency response, $s = j\omega$**

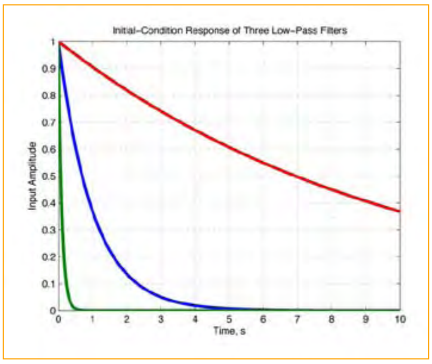
$$x(j\omega) = \frac{a}{(j\omega + a)}u(j\omega)$$



Response of 1st-Order Low-Pass Filters to Step Input and Initial Condition

$$\dot{x}(t) = -ax(t) + au(t)$$
$$a = 0.1,\ 1,\ \text{or } 10$$

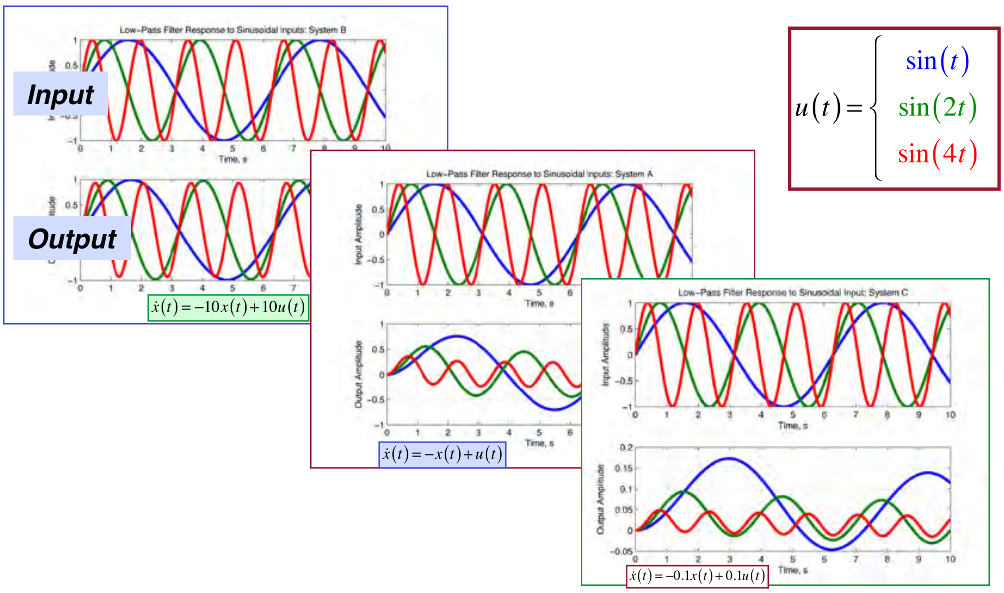




Smoothing effect on sharp changes

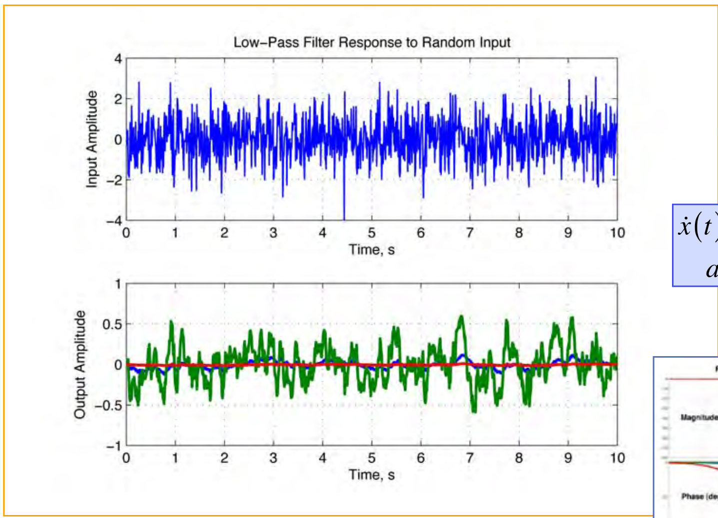
Frequency Response of Dynamic Systems

Response of 1st-Order Low-Pass Filters to Sine-Wave Inputs

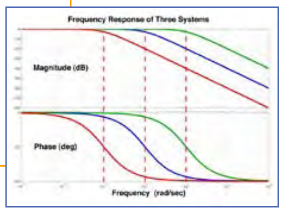


$$u(t) = \begin{cases} \sin(t) \\ \sin(2t) \\ \sin(4t) \end{cases}$$

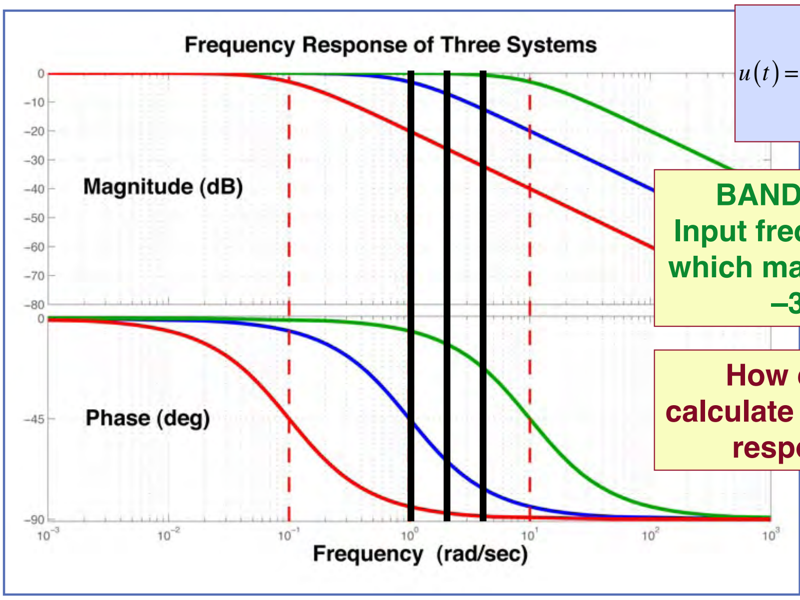
Response of 1st-Order Low-Pass Filters to White Noise



$$\dot{x}(t) = -ax(t) + au(t)$$
$$a = 0.1,\ 1,\ \text{or}\ 10$$



Relationship of Input Frequencies to Filter Bandwidth



$$u(t) = \begin{cases} \sin(\omega t) \\ \sin(2\omega t) \\ \sin(4\omega t) \end{cases}$$

BANDWIDTH
Input frequency at which magnitude = –3dB

How do we calculate frequency response?

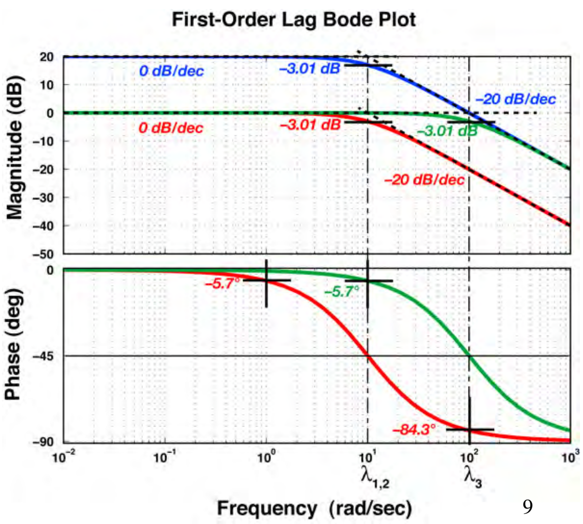
Bode Plot Asymptotes, Departures, and Phase Angles for 1st-Order Lags

- General shape of amplitude ratio governed by **asymptotes**
- Slope of asymptotes changes by **multiples of ±20 dB/dec at poles or zeros**
- Actual AR departs from asymptotes

- AR asymptotes of a real pole
  - When ω = 0, slope = 0 dB/dec
  - When ω ≥ λ, slope = –20 dB/dec

- Phase angle of a real, negative pole
  - When ω = 0, φ = 0°
  - When ω = λ, φ =–45°
  - When ω -> ∞, φ -> –90°

$$x(j\omega)=\frac{a}{(j\omega+a)}u(j\omega)$$



2nd-Order Low-Pass Filter

$$\ddot{x}(t)=-2\zeta\omega_n\dot{x}(t)-\omega_n^2x(t)+\omega_n^2u(t)$$

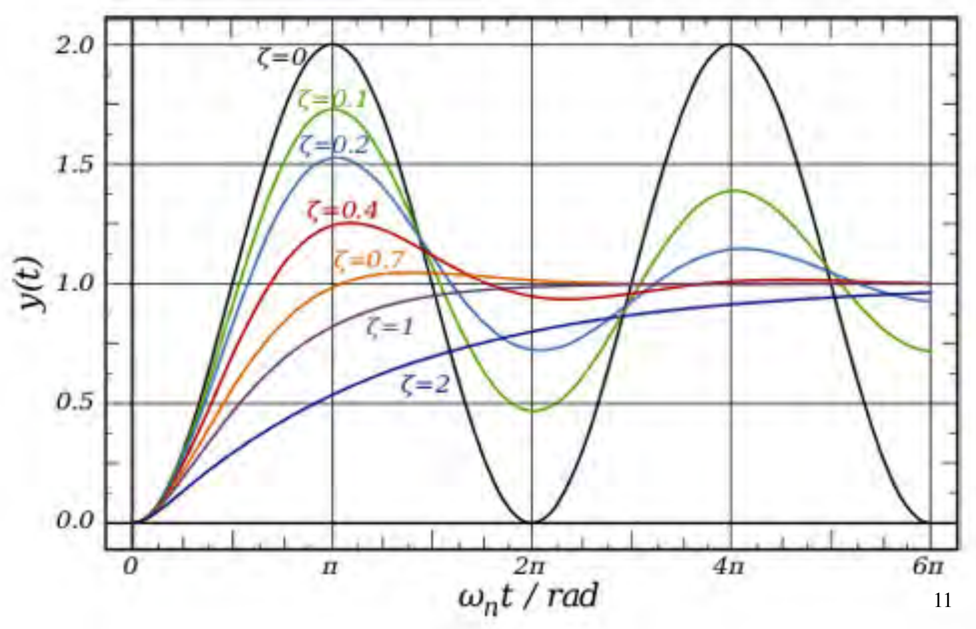
Laplace transform, I.C. = 0

$$x(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}u(s)$$

Frequency response, *s = jω*

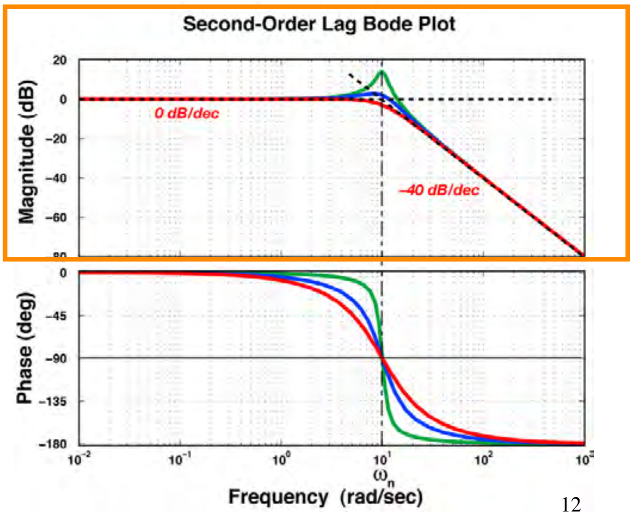
$$x(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2}u(j\omega)$$

2^{nd}-Order Step Response

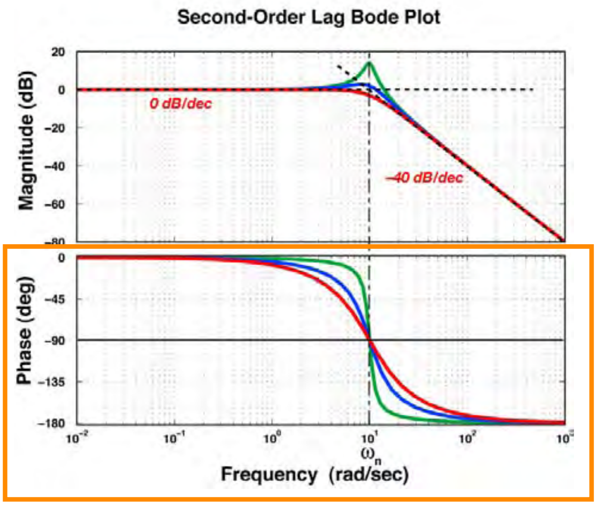


Amplitude Ratio Asymptotes and Departures of Second-Order Bode Plots (No Zeros)

- AR asymptotes of a pair of complex poles
  - When $\omega = 0$, slope = 0 dB/dec
  - When $\omega \geq \omega_n$, slope = –40 dB/ dec
- Height of resonant peak depends on damping ratio



Phase Angles of Second-Order Bode Plots (No Zeros)



- Phase angle of a pair of complex negative poles
  - When $\omega = 0$, $\varphi = 0°$
  - When $\omega = \omega_n$, $\varphi = -90°$
  - When $\omega \to \infty$, $\varphi \to -180°$
- Abruptness of phase shift depends on damping ratio

Transformation of the System Equations

Time-Domain System Equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{H}_\mathbf{x}\mathbf{x}(t) + \mathbf{H}_\mathbf{u}\mathbf{u}(t)$$

Laplace Transforms of System Equations

$$s\mathbf{x}(s) - \mathbf{x}(0) = \mathbf{F}\,\mathbf{x}(s) + \mathbf{G}\,\mathbf{u}(s)$$
$$\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)\right]$$
$$\mathbf{y}(s) = \mathbf{H}_\mathbf{x}\mathbf{x}(s) + \mathbf{H}_\mathbf{u}\mathbf{u}(s)$$

Transfer Function Matrix

Laplace Transform of Output Vector

$$\mathbf{y}(s) = \mathbf{H_x x}(s) + \mathbf{H_u u}(s) = \mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)\right] + \mathbf{H_u u}(s)$$

$$= \left[\mathbf{H_x}\left(s\mathbf{I} - \mathbf{F}\right)^{-1}\mathbf{G} + \mathbf{H_u}\right]\mathbf{u}(s) + \mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}\right]^{-1}\mathbf{x}(0)$$

$$= \textit{Control Effect} + \textit{Initial Condition Effect}$$

Transfer Function Matrix relates control input to system output

with H_u = 0 and neglecting initial condition

$$\boldsymbol{H}(s) = \mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}\right]^{-1}\mathbf{G} \quad (r\ x\ m)$$

Scalar Frequency Response from Transfer Function Matrix

Transfer function matrix with $s = j\omega$

$$\boldsymbol{H}(j\omega) = \mathbf{H_x}\left[j\omega\mathbf{I} - \mathbf{F}\right]^{-1}\mathbf{G} \quad (r\ x\ m)$$

$$\frac{\Delta y_i(s)}{\Delta u_j(s)} = \boldsymbol{H}_{ij}(j\omega) = \mathbf{H}_{\mathbf{x}_i}\left[j\omega\mathbf{I} - \mathbf{F}\right]^{-1}\mathbf{G}_j \quad (r\ x\ m)$$

$$\mathbf{H}_{\mathbf{x}_i} = i^{th} \text{ row of } \mathbf{H_x}$$

$$\mathbf{G}_j = j^{th} \text{ column of } \mathbf{G}$$

Second-Order Transfer Function

Second-order dynamic system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & f_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Second-order transfer function matrix

$$\boldsymbol{H}(s) = \mathbf{H_x A}(s)\mathbf{G} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \frac{\text{adj}\begin{bmatrix} (s - f_{11}) & -f_{12} \\ -f_{21} & (s - f_{22}) \end{bmatrix}}{\det\begin{pmatrix} (s - f_{11}) & -f_{12} \\ -f_{21} & (s - f_{22}) \end{pmatrix}} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & f_{22} \end{bmatrix}$$

$(n = m = r = 2)$

$(r \times n)(n \times n)(n \times m)$
$= (r \times m) = (2 \times 2)$

Scalar Transfer Function from Δu_j to Δy_i

$$H_{ij}(s) = \frac{k_{ij} n_{ij}(s)}{\Delta(s)} = \frac{k_{ij}\left(s^q + b_{q-1}s^{q-1} + \ldots + b_1 s + b_0\right)}{\left(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0\right)}$$

Just <u>one element</u> of the matrix, **H(s)**
Denominator polynomial contains n roots
Each numerator term is a polynomial with q zeros, where q varies from term to term and $\le n - 1$

$$= \frac{k_{ij}(s - z_1)_{ij}(s - z_2)_{ij} \ldots (s - z_q)_{ij}}{(s - \lambda_1)(s - \lambda_2)\ldots(s - \lambda_n)}$$

 # zeros = q
 # poles = n

Scalar Frequency Response Function

Substitute: s = jω

$$H_{ij}(j\omega) = \frac{k_{ij}\left(j\omega - z_1\right)_{ij}\left(j\omega - z_2\right)_{ij}\ldots\left(j\omega - z_q\right)_{ij}}{\left(j\omega - \lambda_1\right)\left(j\omega - \lambda_2\right)\ldots\left(j\omega - \lambda_n\right)}$$

$$= a(\omega) + jb(\omega) \rightarrow AR(\omega)\, e^{j\phi(\omega)}$$

Frequency response is a complex function of input frequency, ω

Real and imaginary parts, or

**** Amplitude ratio and phase angle ****

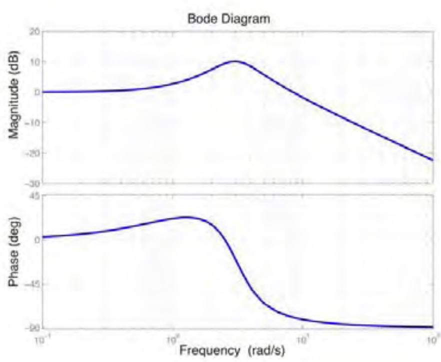
MATLAB Bode Plot with asymp.m

http://www.mathworks.com/matlabcentral/

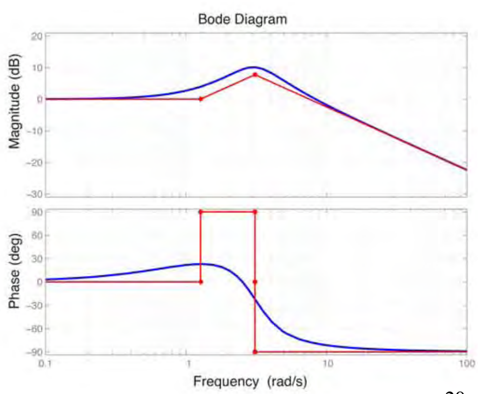
http://www.mathworks.com/matlabcentral/fileexchange/10183-bode-plot-with-asymptotes

2nd-Order Pitch Rate Frequency Response, with zero

bode.m

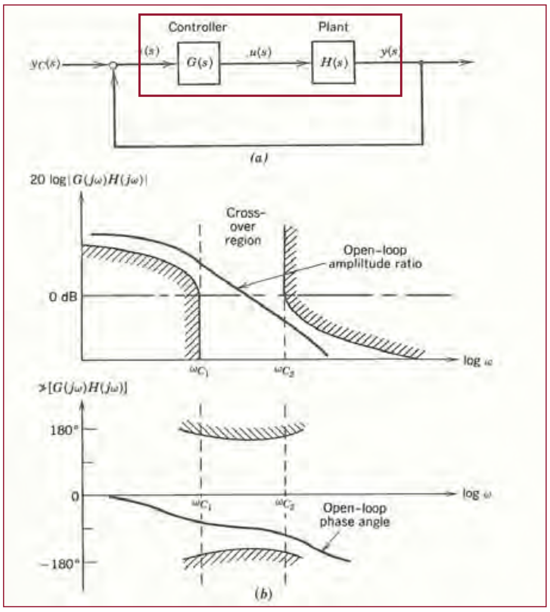


asymp.m



Desirable Open-Loop Frequency Response Characteristics (Bode)

- **High gain (amplitude) at low frequency**
  - Desired response is slowly varying
- **Low gain at high frequency**
  - Random errors vary rapidly
- **Crossover region is problem-specific**

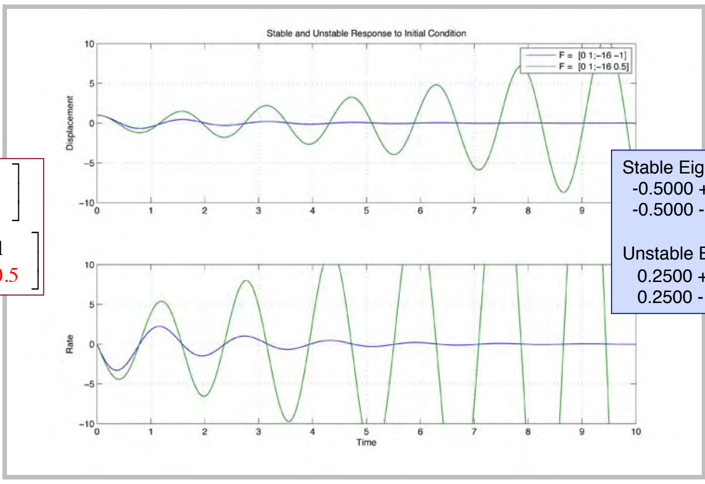


Examples of Proportional LQ Regulator Response

Example: Open-Loop Stable and Unstable 2nd-Order LTI System Response to Initial Condition

$$\mathbf{F}_S = \begin{bmatrix} 0 & 1 \\ -16 & -1 \end{bmatrix}$$

$$\mathbf{F}_U = \begin{bmatrix} 0 & 1 \\ -16 & +0.5 \end{bmatrix}$$



Stable Eigenvalues =
-0.5000 + 3.9686i
-0.5000 - 3.9686i

Unstable Eigenvalues =
0.2500 + 3.9922i
0.2500 - 3.9922i

Example: Stabilizing Effect of Linear-Quadratic Regulators for Unstable 2nd-Order System

$$\min_u J = \min_u \left[\frac{1}{2} \int_0^{\infty} \left(x_1^2 + x_2^2 + r u^2 \right) dt \right]$$

$$u(t) = -\begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -c_1 x_1(t) - c_2 x_2(t)$$

For the unstable system

***r* = 1**
Control Gain (C) =
0.2620 1.0857

Riccati Matrix (S) =
2.2001 0.0291
0.0291 0.1206

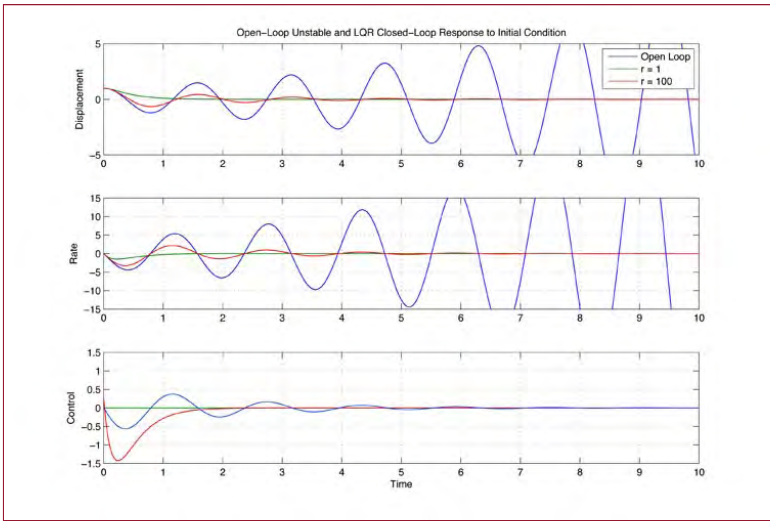
Closed-Loop Eigenvalues =
-6.4061
-2.8656

***r* = 100**
Control Gain (C) =
0.0028 0.1726

Riccati Matrix (S) =
30.7261 0.0312
0.0312 1.9183

Closed-Loop Eigenvalues =
-0.5269 + 3.9683i
-0.5269 - 3.9683i

Example: Stabilizing/Filtering Effect of LQ Regulators for the Unstable 2nd-Order System



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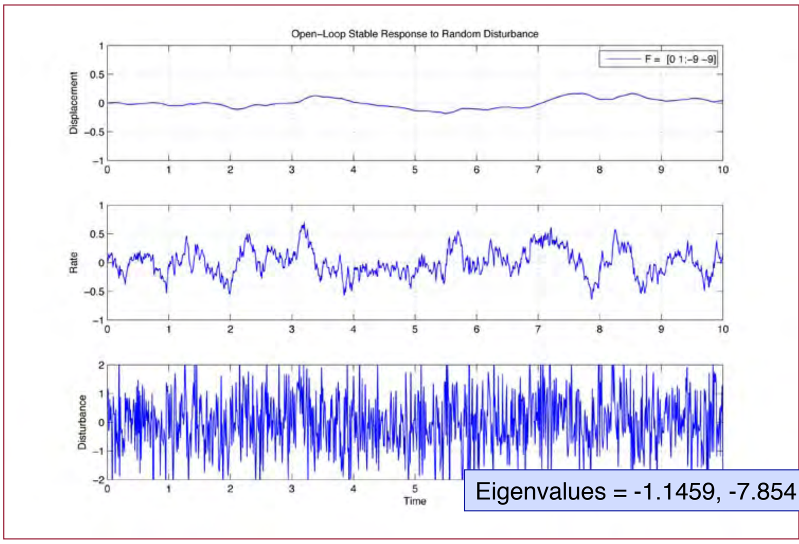
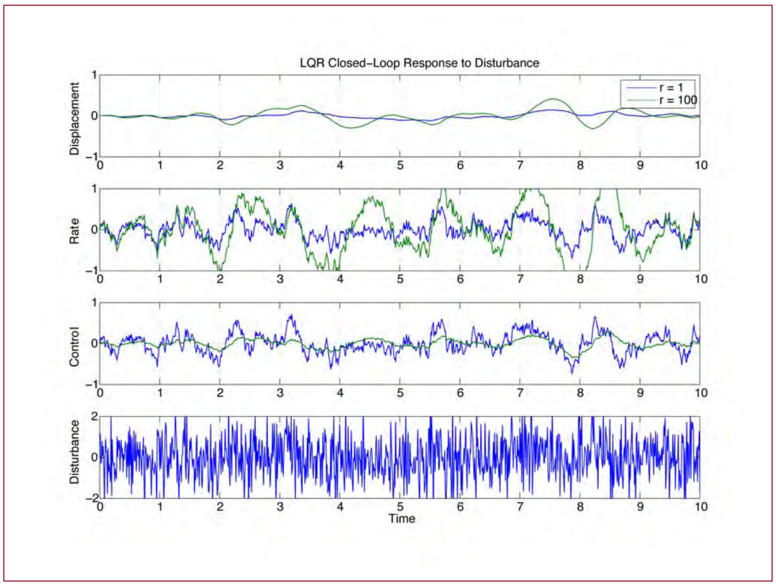
Closed-Loop Eigenvalues =
-0.5269 + 3.9683i
-0.5269 - 3.9683i

Example: Open-Loop Response of the Stable 2nd-Order System to Random Disturbance



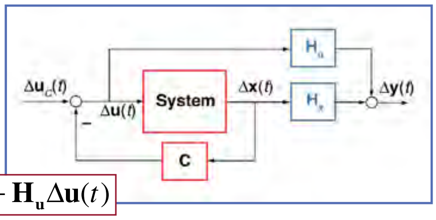
Eigenvalues = -1.1459, -7.8541

Example: Disturbance Response of Unstable System with Two LQRs



LQ Regulators with Output Vector Cost Functions

Quadratic Weighting of the Output



$$\Delta\mathbf{y}(t) = \mathbf{H_x}\Delta\mathbf{x}(t) + \mathbf{H_u}\Delta\mathbf{u}(t)$$

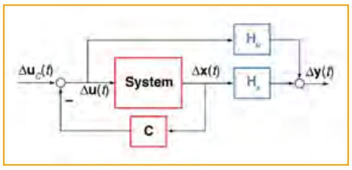
$$J = \frac{1}{2}\int_0^\infty \left[\Delta\mathbf{y}^T(t)\mathbf{Q_y}\Delta\mathbf{y}(t)\right]dt$$
$$= \frac{1}{2}\int_0^\infty \left\{\left[\mathbf{H_x}\Delta\mathbf{x}(t) + \mathbf{H_u}\Delta\mathbf{u}(t)\right]^T \mathbf{Q_y}\left[\mathbf{H_x}\Delta\mathbf{x}(t) + \mathbf{H_u}\Delta\mathbf{u}(t)\right]\right\}dt$$

$$\min_u J = \min_u \frac{1}{2}\int_0^\infty \left\{\begin{bmatrix}\Delta\mathbf{x}^T(t) & \Delta\mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}\mathbf{H_x}^T\mathbf{Q_y}\mathbf{H_x} & \mathbf{H_x}^T\mathbf{Q_y}\mathbf{H_u} \\ \mathbf{H_u}^T\mathbf{Q_y}\mathbf{H_x} & \mathbf{H_u}^T\mathbf{Q_y}\mathbf{H_u} + \mathbf{R}_o\end{bmatrix}\begin{bmatrix}\Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t)\end{bmatrix}\right\}dt$$

$$\min_u J \triangleq \min_u \frac{1}{2}\int_0^\infty \left\{\begin{bmatrix}\Delta\mathbf{x}^T(t) & \Delta\mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}\mathbf{Q}_O & \mathbf{M}_O \\ \mathbf{M}_O^T & \mathbf{R}_O + \mathbf{R}_o\end{bmatrix}\begin{bmatrix}\Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t)\end{bmatrix}\right\}dt$$

$$\Delta\mathbf{u}(t) = \Delta\mathbf{u}_C(t) - \mathbf{C}_O\Delta\mathbf{x}(t)$$

State Rate Can Be Expressed as an Output to be Minimized



$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)$$
$$\Delta\mathbf{y}(t) = \mathbf{H_x}\Delta\mathbf{x}(t) + \mathbf{H_u}\Delta\mathbf{u}(t) \triangleq \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)$$

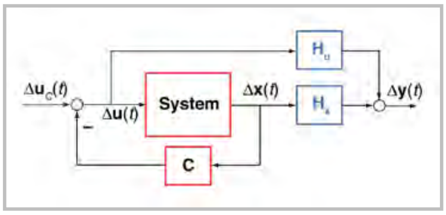
$$J = \frac{1}{2}\int_0^\infty \left[\Delta\mathbf{y}^T(t)\mathbf{Q_y}\Delta\mathbf{y}(t)\right]dt = \frac{1}{2}\int_0^\infty \left[\Delta\dot{\mathbf{x}}^T(t)\mathbf{Q_y}\Delta\dot{\mathbf{x}}(t)\right]dt$$

$$J = \frac{1}{2}\int_0^\infty \left\{\begin{bmatrix}\Delta\mathbf{x}^T(t) & \Delta\mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}\mathbf{F}^T\mathbf{Q_y}\mathbf{F} & \mathbf{F}^T\mathbf{Q_y}\mathbf{G} \\ \mathbf{G}^T\mathbf{Q_y}\mathbf{F} & \mathbf{G}^T\mathbf{Q_y}\mathbf{G} + \mathbf{R}_o\end{bmatrix}\begin{bmatrix}\Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t)\end{bmatrix}\right\}dt$$
$$\triangleq \frac{1}{2}\int_0^\infty \left\{\begin{bmatrix}\Delta\mathbf{x}^T(t) & \Delta\mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}\mathbf{Q}_{SR} & \mathbf{M}_{SR} \\ \mathbf{M}_{SR}^T & \mathbf{R}_{SR} + \mathbf{R}_o\end{bmatrix}\begin{bmatrix}\Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t)\end{bmatrix}\right\}dt$$

$$\Delta\mathbf{u}(t) = \Delta\mathbf{u}_C(t) - \mathbf{C}_{SR}\Delta\mathbf{x}(t)$$

Special case of output weighting

Implicit Model-Following LQ Regulator



Simulator aircraft dynamics

$$\Delta\dot{\mathbf{x}}(t)=\mathbf{F}\Delta\mathbf{x}(t)+\mathbf{G}\Delta\mathbf{u}(t)$$

Ideal aircraft dynamics

$$\Delta\dot{\mathbf{x}}_M(t)=\mathbf{F}_M\Delta\mathbf{x}_M(t)$$

Feedback control law

$$\Delta\mathbf{u}(t)=\Delta\mathbf{u}_C(t)-\mathbf{C}_{IMF}\Delta\mathbf{x}(t)$$

Another special case of output weighting

Implicit Model-Following LQ Regulator

$$\Delta\dot{\mathbf{x}}(t)=\mathbf{F}\Delta\mathbf{x}(t)+\mathbf{G}\Delta\mathbf{u}(t)$$

$$\Delta\dot{\mathbf{x}}_M(t)=\mathbf{F}_M\Delta\mathbf{x}_M(t)$$

If simulation is successful,

$$\Delta\mathbf{x}_M(t)\approx\Delta\mathbf{x}(t)$$

and

$$\Delta\dot{\mathbf{x}}_M(t)\approx\mathbf{F}_M\Delta\mathbf{x}(t)$$

Implicit Model-Following LQ Regulator

Cost function penalizes difference between actual and ideal model dynamics

$$J = \frac{1}{2}\int_0^\infty \left\{ \left[\Delta\dot{\mathbf{x}}(t) - \Delta\dot{\mathbf{x}}_M(t) \right]^T \mathbf{Q}_M \left[\Delta\dot{\mathbf{x}}(t) - \Delta\dot{\mathbf{x}}_M(t) \right] \right\} dt$$

$$J = \frac{1}{2}\int_0^\infty \left\{ \begin{bmatrix} \Delta\mathbf{x}(t) & \Delta\mathbf{u}(t) \end{bmatrix}^T \begin{bmatrix} (\mathbf{F}-\mathbf{F}_M)^T \mathbf{Q}_M (\mathbf{F}-\mathbf{F}_M) & (\mathbf{F}-\mathbf{F}_M)^T \mathbf{Q}_M \mathbf{G} \\ \mathbf{G}^T \mathbf{Q}_M (\mathbf{F}-\mathbf{F}_M) & \mathbf{G}^T \mathbf{Q}_M \mathbf{G} + \mathbf{R}_o \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t) \end{bmatrix} \right\} dt$$

$$\triangleq \frac{1}{2}\int_0^\infty \left\{ \begin{bmatrix} \Delta\mathbf{x}(t) & \Delta\mathbf{u}(t) \end{bmatrix}^T \begin{bmatrix} \mathbf{Q}_{IMF} & \mathbf{M}_{IMF} \\ \mathbf{M}_{IMF}^T & \mathbf{R}_{IMF} + \mathbf{R}_o \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t) \end{bmatrix} \right\} dt$$

Therefore, ideal model is implicit in the optimizing feedback control law

$$\Delta\mathbf{u}(t) = \Delta\mathbf{u}_C(t) - \mathbf{C}_{IMF}\Delta\mathbf{x}(t)$$

Proportional-Derivative Control

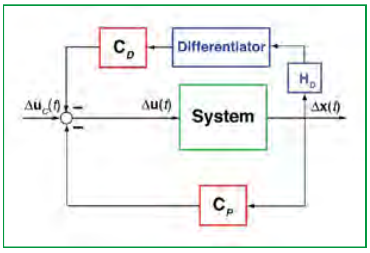
Basic LQ regulators provide proportional control

$$\Delta\mathbf{u}(t) = -\mathbf{C}\Delta\mathbf{x}(t) + \Delta\mathbf{u}_C(t)$$

Derivative feedback can either quicken or slow system response ("lead" or "lag"), depending on the control gain sign

$$\Delta\mathbf{u}(t) = -\mathbf{C}_P\Delta\mathbf{x}(t) - \mathbf{C}_D\Delta\dot{\mathbf{x}}(t) + \Delta\mathbf{u}_C(t)$$

How can proportional-derivative (*PD*) control be implemented with an LQ regulator?



Explicit Proportional-Derivative Control

$$\Delta\mathbf{u}(t) = -\mathbf{C}_P\Delta\mathbf{x}(t) \pm \mathbf{C}_D\Delta\dot{\mathbf{x}}(t) + \Delta\mathbf{u}_C(t)$$

Substitute for the derivative

$$\Delta\mathbf{u}(t) = -\mathbf{C}_P\Delta\mathbf{x}(t) \pm \mathbf{C}_D\left[\mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)\right] + \Delta\mathbf{u}_C(t)$$
$$\left[\mathbf{I} \mp \mathbf{C}_D\mathbf{G}\right]\Delta\mathbf{u}(t) = -\mathbf{C}_P\Delta\mathbf{x}(t) \pm \mathbf{C}_D\mathbf{F}\Delta\mathbf{x}(t) + \Delta\mathbf{u}_C(t)$$

Structure is the same as that of proportional control

$$\Delta\mathbf{u}(t) = \left[\mathbf{I} \mp \mathbf{C}_D\mathbf{G}\right]^{-1}\left[-\left(\mathbf{C}_P \mp \mathbf{C}_D\mathbf{F}\right)\Delta\mathbf{x}(t) + \Delta\mathbf{u}_C(t)\right]$$
$$\triangleq -\mathbf{C}_{PD}\Delta\mathbf{x}(t) + \left[\mathbf{I} \mp \mathbf{C}_D\mathbf{G}\right]^{-1}\Delta\mathbf{u}_C(t)$$

Implement as *ad hoc* modification of proportional LQ control, e.g., $\mathbf{C}_D = \varepsilon\mathbf{C}_{P_{LQ}}$

***Inverse Problem**: Given a stabilizing gain matrix, **C**$_{PD}$, does it minimize some (unknown) cost function? [TBD]*

Implicit Proportional-Derivative Control

Add <u>state rate</u>, i.e., the derivative, to a standard cost function
Include system dynamics in the cost function

$$J = \frac{1}{2}\int_0^\infty \left[\Delta\mathbf{x}^T(t)\mathbf{Q}_\mathbf{x}\Delta\mathbf{x}(t) \pm \Delta\dot{\mathbf{x}}^T(t)\mathbf{Q}_{\dot{\mathbf{x}}}\Delta\dot{\mathbf{x}}(t) + \Delta\mathbf{u}^T(t)\mathbf{R}\Delta\mathbf{u}(t)\right]dt$$

Penalty/reward for fast motions

$$J = \frac{1}{2}\int_0^\infty \left\{\Delta\mathbf{x}^T(t)\mathbf{Q}_\mathbf{x}\Delta\mathbf{x}(t) \pm \left[\mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)\right]^T\mathbf{Q}_{\dot{\mathbf{x}}}\left[\mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)\right] + \Delta\mathbf{u}^T(t)\mathbf{R}\Delta\mathbf{u}(t)\right\}dt$$
$$\triangleq \frac{1}{2}\int_0^\infty \left\{\begin{bmatrix}\Delta\mathbf{x}^T(t) & \Delta\mathbf{u}^T(t)\end{bmatrix}\begin{bmatrix}\mathbf{Q}_{PD} & \mathbf{M}_{PD}\\ \mathbf{M}_{PD}^T & \mathbf{R}_{PD}\end{bmatrix}\begin{bmatrix}\Delta\mathbf{x}(t)\\ \Delta\mathbf{u}(t)\end{bmatrix}\right\}dt$$

Must verify guaranteed stability criteria

$$\Delta\mathbf{u}(t) = -\mathbf{C}_{PD}\Delta\mathbf{x}(t) + \Delta\mathbf{u}_C(t)$$

Cost Functions with Augmented State Vector

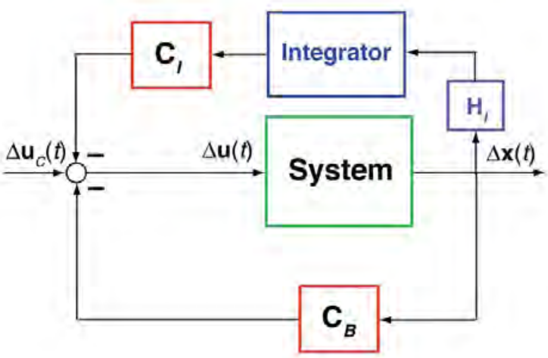
Integral Compensation Can Reduce Steady-State Errors

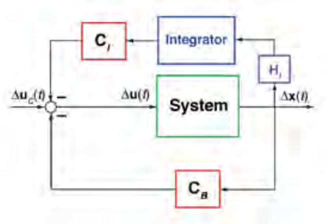
- **Sources of Steady-State Error**
  - **Constant disturbance**
  - **Errors in system dynamic model**

- **Selector matrix, $\mathbf{H}_I$, can reduce or mix integrals in feedback**

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)$$

$$\Delta\dot{\xi}(t) = \mathbf{H}_I\Delta\mathbf{x}(t)$$





LQ Proportional-Integral (*PI*) Control

$$\Delta\mathbf{u}(t) = -\mathbf{C}_B \Delta\mathbf{x}(t) - \mathbf{C}_I \int_0^t \mathbf{H}_I \Delta\mathbf{x}(\tau)\, d\tau$$

$$\triangleq -\mathbf{C}_B \Delta\mathbf{x}(t) - \mathbf{C}_I \Delta\boldsymbol{\xi}(t) + \Delta\mathbf{u}_C(t)$$

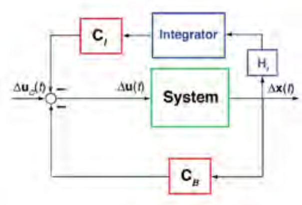
where the integral state is

$$\boldsymbol{\xi}(t) \triangleq \int_0^t \mathbf{H}_I \Delta\mathbf{x}(\tau)\, d\tau$$

$$\dim(\mathbf{H}_I) = m \times n$$

define $\chi(t) \triangleq \begin{bmatrix} \Delta\mathbf{x}(t) \\ \boldsymbol{\xi}(t) \end{bmatrix}$

Integral State is Added to the Cost Function and the Dynamic Model



$$\min_{\Delta\mathbf{u}} J = \frac{1}{2}\int_0^\infty \left[\Delta\mathbf{x}^T(t)\mathbf{Q}_\mathbf{x}\Delta\mathbf{x}(t) + \Delta\boldsymbol{\xi}^T(t)\mathbf{Q}_\xi\Delta\boldsymbol{\xi}(t) + \Delta\mathbf{u}^T(t)\mathbf{R}\Delta\mathbf{u}(t) \right] dt$$

$$= \frac{1}{2}\int_0^\infty \left[\Delta\boldsymbol{\chi}^T(t) \begin{bmatrix} \mathbf{Q}_\mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_\xi \end{bmatrix} \Delta\boldsymbol{\chi}(t) + \Delta\mathbf{u}^T(t)\mathbf{R}\Delta\mathbf{u}(t) \right] dt$$

subject to $\quad \Delta\dot{\boldsymbol{\chi}}(t) = \mathbf{F}_\chi \Delta\boldsymbol{\chi}(t) + \mathbf{G}_\chi \Delta\mathbf{u}(t)$

$$\Delta\mathbf{u}(t) = -\mathbf{C}_\chi \Delta\boldsymbol{\chi}(t) + \Delta\mathbf{u}_C(t)$$

$$= -\mathbf{C}_B \Delta\mathbf{x}(t) - \mathbf{C}_I \Delta\boldsymbol{\xi}(t) + \Delta\mathbf{u}_C(t)$$

Integral State is Added to the Cost Function and the Dynamic Model

$$\Delta\mathbf{u}(t) = -\mathbf{C}_\chi \Delta\boldsymbol{\chi}(t) + \Delta\mathbf{u}_C(t)$$
$$= -\mathbf{C}_B \Delta\mathbf{x}(t) - \mathbf{C}_I \Delta\boldsymbol{\xi}(t) + \Delta\mathbf{u}_C(t)$$

$$\Delta\mathbf{u}(s) = -\mathbf{C}_\chi \Delta\boldsymbol{\chi}(s) + \Delta\mathbf{u}_C(s)$$
$$= -\mathbf{C}_B \Delta\mathbf{x}(s) - \mathbf{C}_I \Delta\boldsymbol{\xi}(s) + \Delta\mathbf{u}_C(s)$$
$$= -\mathbf{C}_B \Delta\mathbf{x}(s) - \mathbf{C}_I \frac{\mathbf{H}_\mathbf{x}\Delta\mathbf{x}(s)}{s} + \Delta\mathbf{u}_C(s)$$

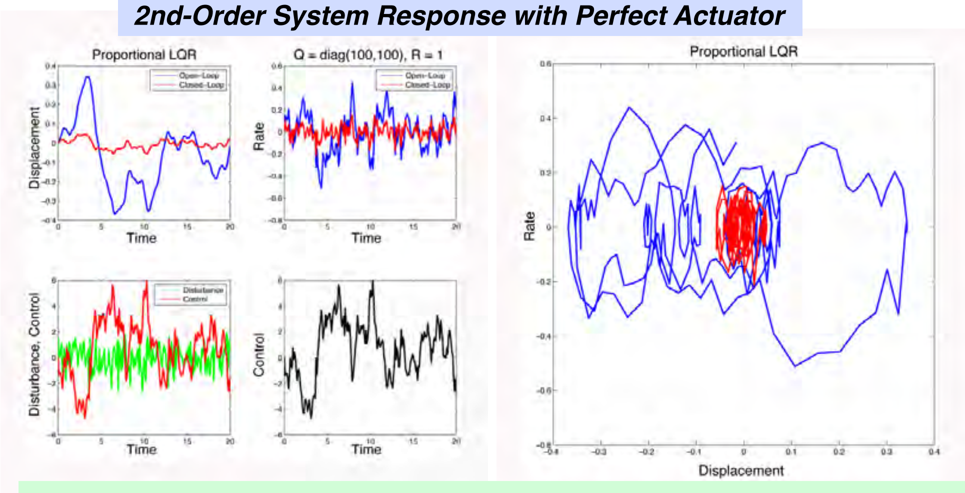
$$\Delta\mathbf{u}(s) = -\frac{\mathbf{C}_B s\Delta\mathbf{x}(s) + \mathbf{C}_I \mathbf{H}_\mathbf{x}\Delta\mathbf{x}(s)}{s} + \Delta\mathbf{u}_C(s)$$
$$= -\frac{\left[\mathbf{C}_B s + \mathbf{C}_I \mathbf{H}_\mathbf{x}\right]}{s}\Delta\mathbf{x}(s) + \Delta\mathbf{u}_C(s)$$

Form of (m x n) Bode Plots from **Δx** *to* **Δu***?*

Actuator Dynamics and Proportional-Filter LQ Regulators

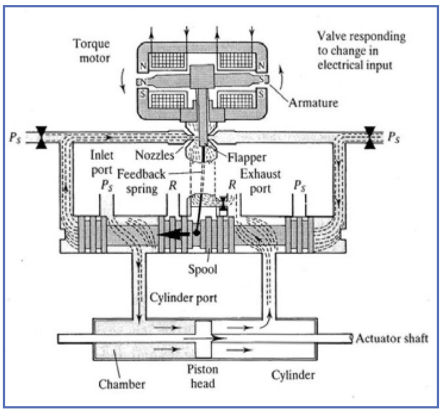
Proportional LQ Regulator: High-Frequency Control in Response to High-Frequency Disturbances

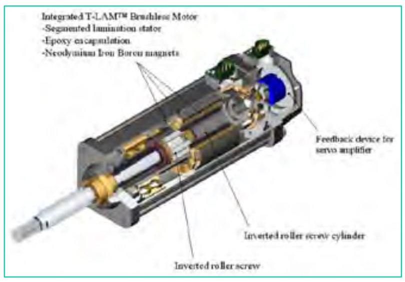
2nd-Order System Response with Perfect Actuator



Good disturbance rejection, but may high bandwidth may be unrealistic

Actuator Dynamics May Impact System Response

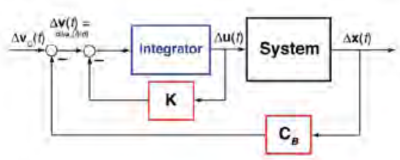




Actuator Dynamics May Affect System Response

Augment state dynamics to include actuator dynamics

$$\begin{bmatrix} \Delta\dot{\mathbf{x}}(t) \\ \Delta\dot{\mathbf{u}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \Delta\mathbf{v}(t)$$



Control variable is actuator forcing function

$$\Delta\mathbf{v}(t) = \Delta\dot{\mathbf{u}}_{Integrator}(t) = -\mathbf{C}_B\Delta\mathbf{x}(t) + \Delta\mathbf{v}_C(t) \text{ is } \textbf{sub-optimal}$$

LQ Regulator with Actuator Dynamics

Cost function is minimized with re-defined state and control vectors

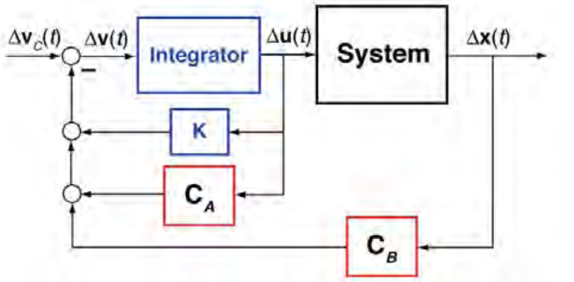
$$\Delta\boldsymbol{\chi}(t) = \begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t) \end{bmatrix}; \quad \mathbf{F}_\chi = \begin{bmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix}; \quad \mathbf{G}_\chi = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

$$\min_{\Delta\mathbf{u}} J = \frac{1}{2}\int_0^\infty \left[\Delta\mathbf{x}^T(t)\mathbf{Q}_\mathbf{x}\Delta\mathbf{x}(t) + \Delta\mathbf{u}^T(t)\mathbf{R}_\mathbf{u}\Delta\mathbf{u}(t) + \Delta\mathbf{v}^T(t)\mathbf{R}_\mathbf{v}\Delta\mathbf{v}(t) \right] dt$$

$$= \frac{1}{2}\int_0^\infty \left[\Delta\boldsymbol{\chi}^T(t) \begin{bmatrix} \mathbf{Q}_\mathbf{x} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_\mathbf{u} \end{bmatrix} \Delta\boldsymbol{\chi}(t) + \Delta\mathbf{v}^T(t)\mathbf{R}_\mathbf{v}\Delta\mathbf{v}(t) \right] dt$$

$$\text{subject to} \quad \Delta\dot{\boldsymbol{\chi}}(t) = \mathbf{F}_\chi\Delta\boldsymbol{\chi}(t) + \mathbf{G}_\chi\Delta\mathbf{v}(t)$$

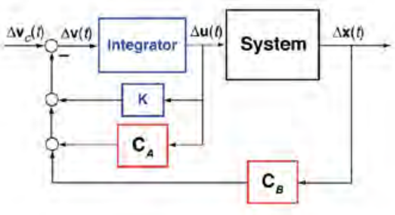
LQ Regulator with Actuator Dynamics



$$\Delta\mathbf{v}(t) = -\mathbf{C}_\chi \Delta\boldsymbol{\chi}(t) + \Delta\mathbf{v}_C(t)$$
$$= -\mathbf{C}_B \Delta\mathbf{x}(t) - \mathbf{C}_A \Delta\mathbf{u}(t) + \Delta\mathbf{v}_C(t)$$

$$\Delta\mathbf{v}(s) = -\mathbf{C}_B \Delta\mathbf{x}(s) - \mathbf{C}_A \Delta\mathbf{u}(s) + \Delta\mathbf{v}_C(s)$$

LQ Regulator with Actuator Dynamics



$$\Delta\dot{\mathbf{u}}(t) = -\mathbf{K}\Delta\mathbf{u}(t) - \mathbf{C}_A \Delta\mathbf{u}(t) - \mathbf{C}_B \Delta\mathbf{x}(t) + \Delta\mathbf{v}_C(t)$$
$$s\Delta\mathbf{u}(s) = -\mathbf{K}\Delta\mathbf{u}(s) - \mathbf{C}_A \Delta\mathbf{u}(s) - \mathbf{C}_B \Delta\mathbf{x}(s) + \Delta\mathbf{v}_C(s) + \Delta\mathbf{u}(0)$$

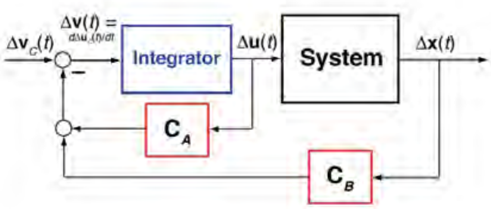
Control Displacement

$$\left[s\mathbf{I} + \mathbf{K} + \mathbf{C}_A\right]\Delta\mathbf{u}(s) = -\mathbf{C}_B \Delta\mathbf{x}(s) + \Delta\mathbf{v}_C(s)$$
$$\Delta\mathbf{u}(s) = \left[s\mathbf{I} + \mathbf{K} + \mathbf{C}_A\right]^{-1}\left[-\mathbf{C}_B \Delta\mathbf{x}(s) + \Delta\mathbf{v}_C(s)\right]$$

LQ Regulator with Artificial Actuator Dynamics

LQ control variable is derivative of actual system control

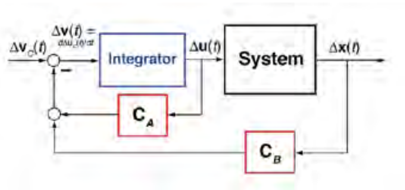
$$\begin{bmatrix} \Delta\dot{\mathbf{x}}(t) \\ \Delta\dot{\mathbf{u}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \Delta\mathbf{v}(t)$$



$$\Delta\mathbf{v}(t) = \Delta\dot{\mathbf{u}}_{Int}(t) = -\mathbf{C}_B\Delta\mathbf{x}(t) - \mathbf{C}_A\Delta\mathbf{u}(t) + \Delta\mathbf{v}_C(t)$$

C_A introduces artificial actuator dynamics

Proportional-Filter (PF) LQ Regulator



$$\Delta\boldsymbol{\chi}(t) = \begin{bmatrix} \Delta\mathbf{x}(t) \\ \Delta\mathbf{u}(t) \end{bmatrix}; \quad \mathbf{F}_\chi = \begin{bmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \quad \mathbf{G}_\chi = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}$$

Optimal LQ Regulator

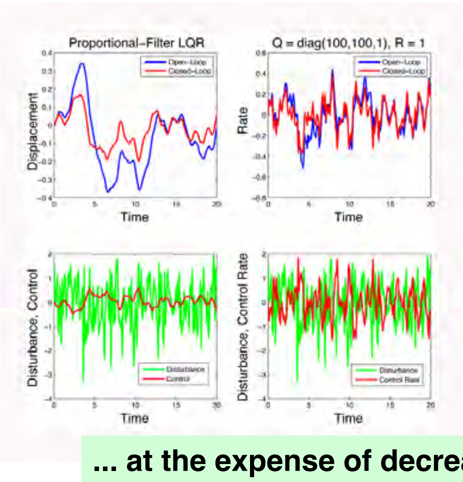
$$\Delta\mathbf{v}(t) = \Delta\dot{\mathbf{u}}_{Integrator}(t) = -\mathbf{C}_\chi\Delta\boldsymbol{\chi}(t) + \Delta\mathbf{v}_C(t)$$

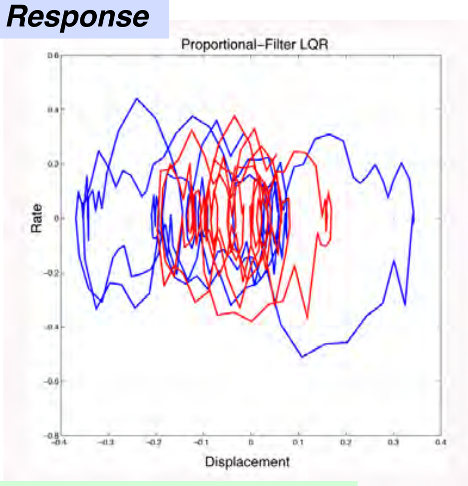
C_A provides *low-pass filtering effect* on the control input

$$\Delta\mathbf{u}(s) = \left[s\mathbf{I} + \mathbf{C}_A\right]^{-1}\left[-\mathbf{C}_B\Delta\mathbf{x}(s) + \Delta\mathbf{v}_C(s)\right]$$

Proportional-Filter LQ Regulator Reduces High-Frequency Control Signals

2nd-Order System Response





... at the expense of decreased disturbance rejection

Next Time: Linear-Quadratic Control System Design

Supplemental Material

Implicit Model-Following Linear-Quadratic Regulator

Model the response of one airplane with another using feedback control

Princeton Variable-Response Research Aircraft (*VRA*)