Principles for Optimal Control, Part 2

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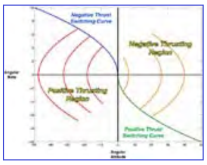
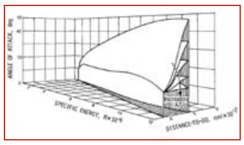
Optimal Control and Estimation, MAE 546, Princeton University, 2015

- Minimum Principle
- Hamilton-Jacobi-Bellman Equation (Dynamic Programming)
- Terminal State Equality Constraint
- Linear, Time-Invariant, Minimum-Time Control Problem

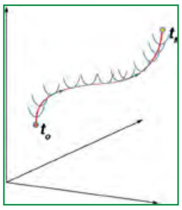




http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

The Minimum Principle

The Minimum Principle*



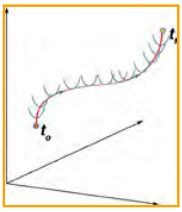
Variational necessary and sufficient conditions imply that minimum *H* is optimal

$$\frac{\partial H[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t]}{\partial \mathbf{u}} = \mathbf{0} \quad \text{in } \left(t_0, t_f\right)$$

$$\frac{\partial^2 H[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t),t]}{\partial \mathbf{u}^2} > 0 \text{ in } \left(t_0, t_f\right)$$

$$H^* = H\left[\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t)\right] \le H\left[\mathbf{x}^*(t),\mathbf{u}(t),\boldsymbol{\lambda}^*(t)\right]$$

** After the "Maximum Principle" of Pontryagin, et al, 1950s (opposite convention for sign of Hamiltonian)*



Control Perturbation Can Only Increase Cost

Effect of control perturbation on optimal *H* and *J* *

$$J\left[\mathbf{u}^*(t)+\Delta\mathbf{u}(t)\right] - J\left[\mathbf{u}^*(t)\right] = \phi\left[\mathbf{x}^*(t_f)\right] - \phi\left[\mathbf{x}^*(t_f)\right]$$
$$+\int_{t_o}^{t_f} \left\langle \left\{ H\left[\mathbf{x}^*(t),\mathbf{u}^*(t)+\Delta\mathbf{u}(t),\boldsymbol{\lambda}^*(t)\right] - \boldsymbol{\lambda}^{*T}(t)\dot{\mathbf{x}}^*(t) \right\} - \left\{ H\left[\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t)\right] - \boldsymbol{\lambda}^{*T}(t)\dot{\mathbf{x}}^*(t) \right\} \right\rangle dt$$

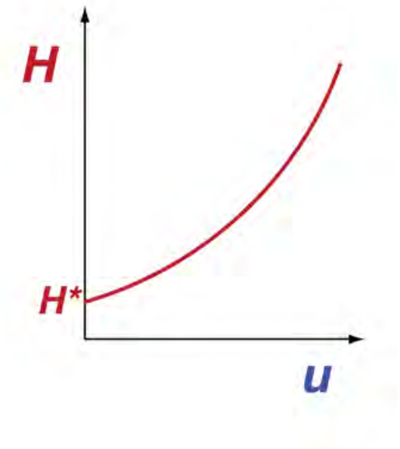
Control perturbation has no effect on terminal cost or $\boldsymbol{\lambda}^T \partial\mathbf{x}/\partial t$

$$J\left[\mathbf{u}^*(t)+\Delta\mathbf{u}(t)\right] - J\left[\mathbf{u}^*(t)\right] =$$
$$\int_{t_o}^{t_f} \left\langle \left\{ H\left[\mathbf{x}^*(t),\mathbf{u}^*(t)+\Delta\mathbf{u}(t),\boldsymbol{\lambda}^*(t)\right] \right\} - \left\{ H\left[\mathbf{x}^*(t),\mathbf{u}^*(t),\boldsymbol{\lambda}^*(t)\right] \right\} \right\rangle dt \ge 0$$

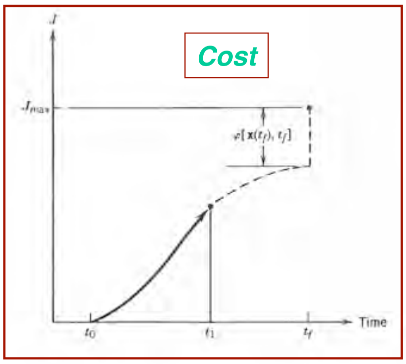
Assuming that $\mathbf{x}^*(t)$ and $\boldsymbol{\lambda}^*(t)$ are the optimal values

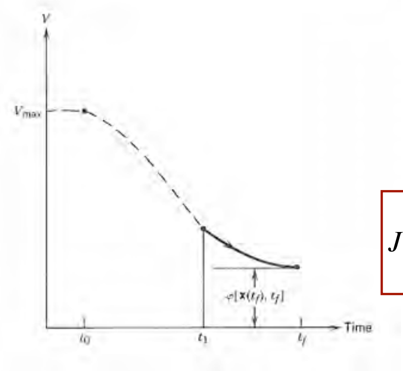
Application of the Minimum Principle with Bounded Control

- **Minimum principle applies**
  - when control is limited such that $\partial H/\partial u \neq 0$
  - in some cases of singular control, e.g. "bang-bang control" (*TBD*)



Dynamic Programming





Cost Function vs. Value Function

Optimal Cost Function (i.e., accrued cost) at t_1

$$J^*(t_1) = \int_{t_o}^{t_1} L[\mathbf{x}(\tau), \mathbf{u}(\tau)]\, d\tau$$

Optimal Cost Function at t_f

$$J^*(t_f) = \phi[\mathbf{x}(t_f)] + \int_{t_o}^{t_f} L[\mathbf{x}(\tau), \mathbf{u}(\tau)]\, d\tau \triangleq J^*_{\max}$$

Cost Function vs. Value Function

Optimal Value Function (i.e., remaining cost) at t_1

$$V^*(x_1, t_1) = \phi[\mathbf{x}^*(t_f)] + \int_{t_1}^{t_f} L[\mathbf{x}^*(\tau), \mathbf{u}^*(\tau)]\, d\tau$$

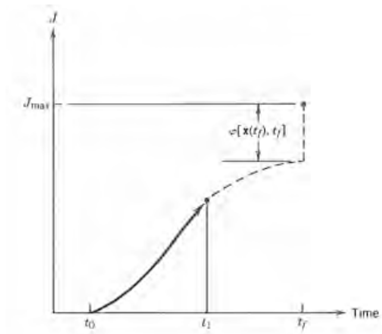
$$= \phi[\mathbf{x}^*(t_f)] - \int_{t_f}^{t_1} L[\mathbf{x}^*(\tau), \mathbf{u}^*(\tau)]\, d\tau$$

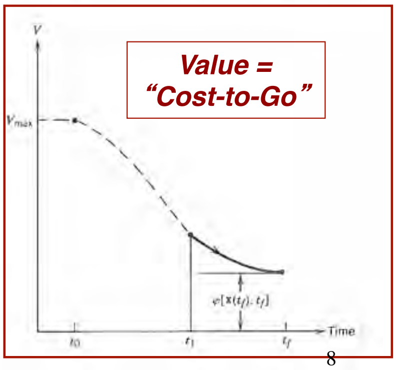
$$= \min_{\mathbf{u}} \left\{ \phi[\mathbf{x}^*(t_f)] - \int_{t_f}^{t_1} L[\mathbf{x}^*(\tau), \mathbf{u}(\tau)]\, d\tau \right\}$$

Optimal Value Function at t_o

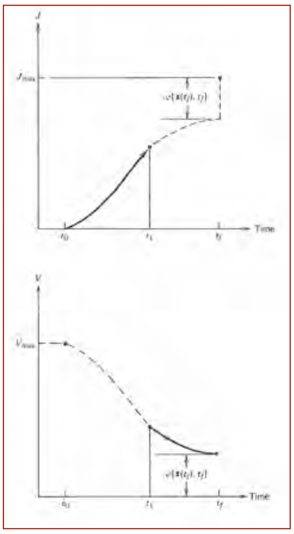
$$V^*(x_o, t_o) = \phi[\mathbf{x}^*(t_f)] - \int_{t_f}^{t_0} L[\mathbf{x}^*(\tau), \mathbf{u}^*(\tau)]\, d\tau$$

$$\triangleq V^*_{\max} = J^*_{\max}$$





Time Derivative of the Value Function



Optimal Value Function at t_1

$$V^*\left(x_1,t_1\right)=\phi\left[\mathbf{x}^*(t_f)\right]-\int_{t_f}^{t_1} L\left[\mathbf{x}^*(\tau),\mathbf{u}^*(\tau)\right]d\tau$$

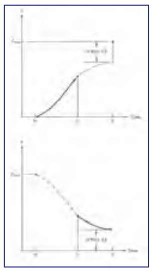
- **Total time-derivative of V^***
  - Rate at which Value is spent
  - Integrand of Value function

$$\left.\frac{dV^*}{dt}\right|_{t=t_1}=-L\left[\mathbf{x}^*(t_1),\mathbf{u}^*(t_1)\right]$$

$$=\left.\left(\frac{\partial V^*}{\partial t}+\frac{\partial V^*}{\partial \mathbf{x}}\dot{\mathbf{x}}+\frac{\partial V^*}{\partial \mathbf{u}}\dot{\mathbf{u}}\right)\right|_{t=t_1}$$

= 0 on optimal trajectory

Dynamic Programming: Hamilton-Jacobi-Bellman Equation



Rearrange to solve for partial derivative wrt t

$$\left.\frac{\partial V^*}{\partial t}\right|_{t=t_1}=\left.\left(\frac{dV^*}{dt}-\frac{\partial V^*}{\partial \mathbf{x}}\dot{\mathbf{x}}\right)\right|_{t=t_1}=\left.\left(-L\left[\mathbf{x}^*,\mathbf{u}^*\right]-\frac{\partial V^*}{\partial \mathbf{x}}\dot{\mathbf{x}}\right)\right|_{t=t_1}$$

$$=\left.\left(-L\left[\mathbf{x}^*,\mathbf{u}^*\right]-\frac{\partial V^*}{\partial \mathbf{x}}\mathbf{f}\left[\mathbf{x}^*,\mathbf{u}^*\right]\right)\right|_{t=t_1}$$

Define a Hamiltonian for the system

$$\left.\frac{\partial V^*}{\partial t}\right|_{t=t_1}\triangleq -H\left\{\mathbf{x}^*(t_1),\mathbf{u}^*(t_1),\frac{\partial V^*}{\partial \mathbf{x}}(t_1)\right\}$$

$$\triangleq -\min_{\mathbf{u}} H\left\{\mathbf{x}^*(t_1),\mathbf{u}(t_1),\frac{\partial V^*}{\partial \mathbf{x}}(t_1)\right\}\quad in\left[t_o,t_f\right]$$

Principle of Optimality (Bellman, 1957)

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

HJB equation **is a partial differential equation**

$$\left.\frac{\partial V^*}{\partial t}\right|_{t=t_1} \triangleq -H\left\{\mathbf{x}^*(t_1),\mathbf{u}^*(t_1),\frac{\partial V^*}{\partial \mathbf{x}}(t_1)\right\}$$
$$\triangleq -\min_{\mathbf{u}} H\left\{\mathbf{x}^*(t_1),\mathbf{u}(t_1),\frac{\partial V^*}{\partial \mathbf{x}}(t_1)\right\} \quad in\left[t_o,t_f\right]$$

Boundary condition

$$V^*\left[\mathbf{x}^*(t_f)\right] = \phi\left[\mathbf{x}^*(t_f)\right]$$

Necessary and Sufficient Condition for Optimality

$$\left.\frac{\partial V^*}{\partial t}\right|_{t=t_1} = -\min_{\mathbf{u}(t)} H\left\{\mathbf{x}^*(t_1),\mathbf{u}(t_1),\frac{\partial V^*}{\partial \mathbf{x}}(t_1)\right\}$$

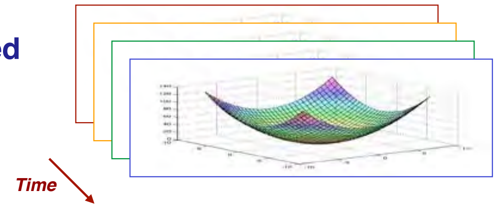
Minimum of *H* w.r.t. u(*t*) requires stationarity and convexity

$V^*[x(t),t]$ is a Hypersurface That Defines Minimum Cost Control

- $V^*[x(t),t]$ is the integral of the HJB equation
  - V^* is a scalar function of the state
- Ideally, the time-varying hypersurface of V^* is bowl-shaped
- Minimum of hypersurface specifies optimal control policy

At the terminal time

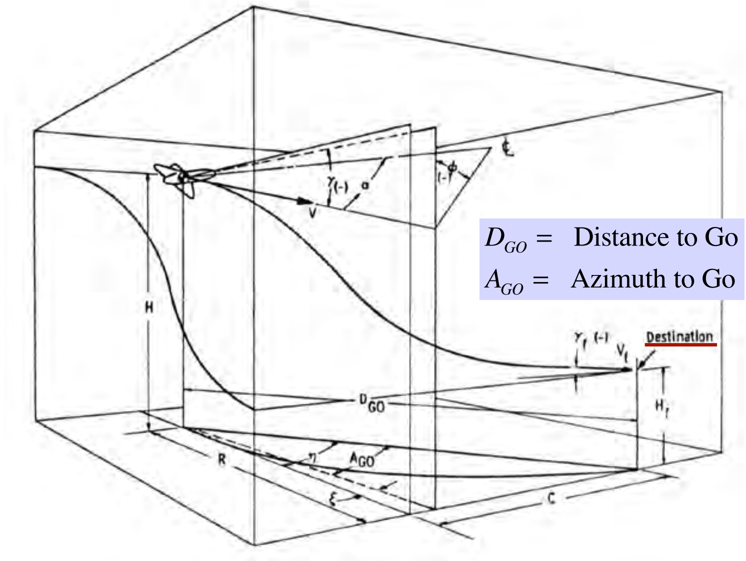
$$V^*\left[\mathbf{x}^*(t_f)\right] = \phi\left[\mathbf{x}^*(t_f)\right]$$



$$\mathbf{u}^*(t) = \mathbf{u}^*\left\{V^*\left[\mathbf{x}^*(t)\right]\right\}$$

Space Shuttle Reentry Example

Optimal Guidance for Space Shuttle Reentry

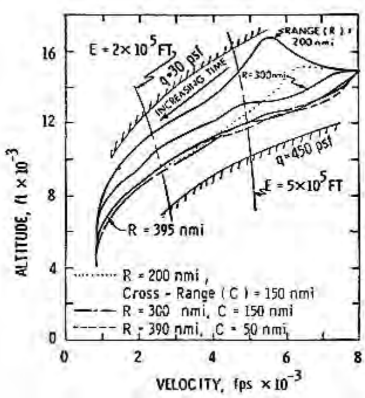


D_{GO} = Distance to Go

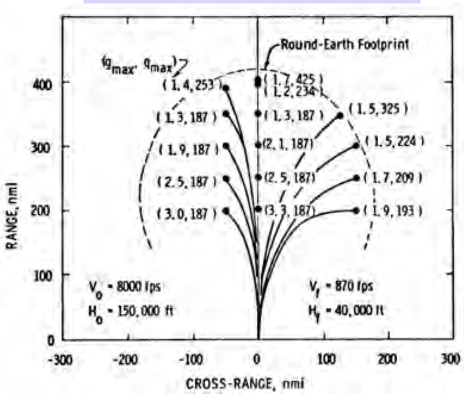
A_{GO} = Azimuth to Go

Optimal Trajectories for Space Shuttle Reentry

Altitude vs. Velocity



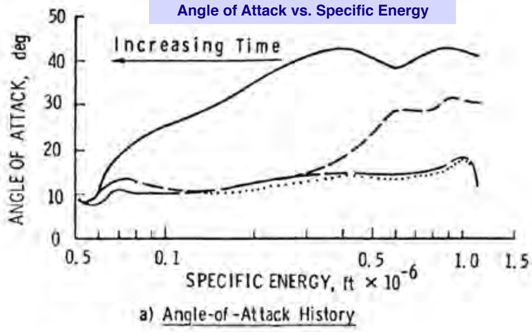
Range vs. Cross-Range ("footprint")



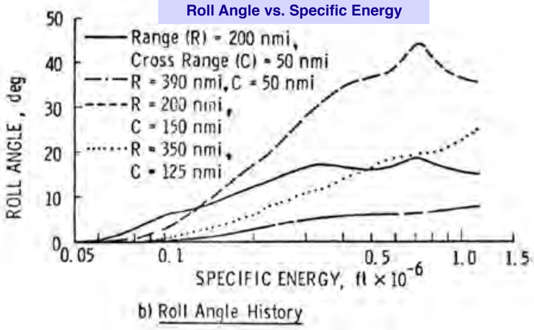
Numerical solutions using steepest-descent and conjugate-gradient algorithms

Optimal Controls for Space Shuttle Reentry

Angle of Attack vs. Specific Energy



Roll Angle vs. Specific Energy



- Independent variable is specific total energy rather than time
- On reentry, total energy decreases as time increases

Optimal Guidance System Derived from Optimal Trajectories

Angle of Attack and Roll Angle vs. Specific Energy

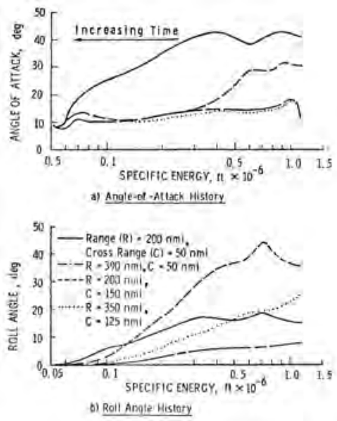
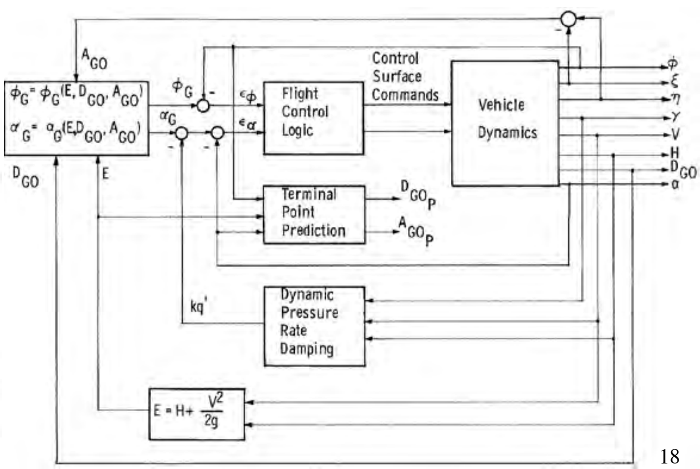
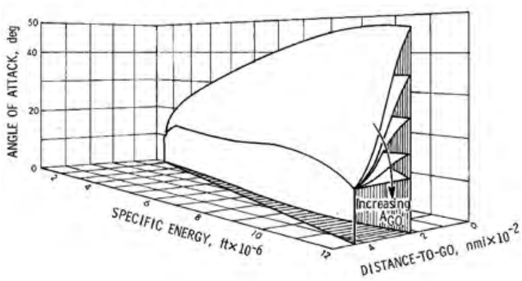


Diagram of Energy-Guidance Law

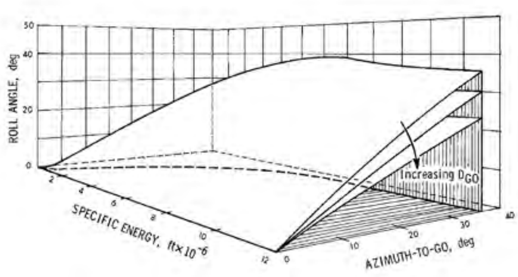


Guidance Functions for Space Shuttle Reentry

Angle of Attack Guidance Function



Roll Angle Guidance Function



- Guidance surfaces can be implemented with
  - Table lookup
  - Computational neural networks

Relationship of HJB Equation to Other Principles of Optimality

$$\left.\frac{\partial V^*}{\partial t}\right|_{t=t_1} \triangleq -H\left\{\mathbf{x}^*(t_1),\mathbf{u}^*(t_1),\frac{\partial V^*}{\partial \mathbf{x}}(t_1)\right\}$$

$$\triangleq -\min_{\mathbf{u}} H\left\{\mathbf{x}^*(t_1),\mathbf{u}(t_1),\frac{\partial V^*}{\partial \mathbf{x}}(t_1)\right\} \quad in\left[t_o,t_f\right]$$

Calculus of Variations (Euler-Lagrange Equations)

$$\frac{\partial V^*}{\partial \mathbf{x}}(t_1) = \boldsymbol{\lambda}^T(t_1)$$

Minimum Principle

$$\min_{\mathbf{u}} H\left\{\mathbf{x}^*(t_1),\mathbf{u}(t_1),\frac{\partial V^*}{\partial \mathbf{x}}(t_1)\right\} \quad in\left[t_o,t_f\right]$$ **defines optimality**

Terminal State Equality Constraint

Minimize a Cost Function Subject to a Terminal State Equality Constraint

$$\min_{\mathbf{u}(t)} J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t), \mathbf{u}(t)\right] dt$$

- **subject to**

Dynamic Constraint

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)], \quad x(t_o) \text{ given}$$

Terminal State Equality Constraint

$$\psi\left[\mathbf{x}(t_f)\right] \equiv 0 \quad \text{(scalar)}$$

Terminal State Equality Constraints

Soft Constraint

$$\min_{\mathbf{u}(t)} \phi\left[\mathbf{x}(t_f)\right]$$

$$\phi\left[\mathbf{x}(t_f)\right] \simeq 0 \quad \text{is OK}$$

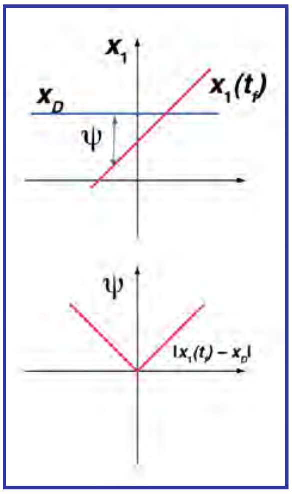
$$\psi\left[\mathbf{x}(t_f)\right] = x_{1_f} - x_D \equiv 0$$

Hard Constraint

$$\psi\left[\mathbf{x}(t_f)\right] \equiv 0$$

$$\psi\left[\mathbf{x}(t_f)\right] = \left|x_{1_f} - x_D\right| \equiv 0$$

Examples



Cost Function Augmented by Terminal State Equality Constraint

$$J_{Constrained} = J_{Unconstrained} + \mu\psi\left[\mathbf{x}(t_f)\right]$$
$$\triangleq J_0 + \mu J_1$$

$\mu =$ constant scalar Lagrange multiplier for terminal constraint

- Separate solution into two parts
  - Optimize original cost function alone
  - Optimize for constraint alone

Euler-Lagrange Equations and 1st Variation for Unconstrained Optimization

$$\boldsymbol{\lambda}_0(t_f) = \left\{ \frac{\partial \phi[\mathbf{x}(t_f)]}{\partial \mathbf{x}} \right\}^T$$

$$\dot{\boldsymbol{\lambda}}_0 = -\left\{ \frac{\partial H_0[\mathbf{x},\mathbf{u},\boldsymbol{\lambda}_0,t]}{\partial \mathbf{x}} \right\}^T = -\left[\frac{\partial L}{\partial \mathbf{x}} + \boldsymbol{\lambda}_0^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]^T = -\left[L_{\mathbf{x}} + \mathbf{F}^T \boldsymbol{\lambda}_0 \right]$$

Assuming that these equations are satisfied, the first variation is

$$\Delta J_0 = \int_{t_o}^{t_f} \left(\frac{\partial H_0}{\partial \mathbf{u}} \Delta \mathbf{u} \right) dt = \int_{t_o}^{t_f} \left(L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G} \right) \Delta \mathbf{u} \, dt$$

Terminal Constraint "Cost" Augmented by Dynamic Constraint

$$J_1 = \psi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} \left\{ \boldsymbol{\lambda}_1^T(t) \left[\mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)] - \dot{\mathbf{x}}(t) \right] \right\} dt$$

$$= \psi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} \left\{ \boldsymbol{\lambda}_1^T \mathbf{f}[\mathbf{x},\mathbf{u}] - \boldsymbol{\lambda}_1^T \dot{\mathbf{x}} \right\} dt$$

$$J_1 \triangleq \psi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} \left\{ H_1[\mathbf{x},\mathbf{u},\boldsymbol{\lambda}_1^T] - \boldsymbol{\lambda}_1^T \dot{\mathbf{x}} \right\} dt$$

$$= \psi\left[\mathbf{x}(t_f)\right] + \left[\boldsymbol{\lambda}_1^T(t_0)\mathbf{x}(t_0) - \boldsymbol{\lambda}_1^T(t_f)\mathbf{x}(t_f) \right]$$

$$+ \int_{t_o}^{t_f} \left\{ H_1\left[\mathbf{x},\mathbf{u},\boldsymbol{\lambda}_1\right] + \dot{\boldsymbol{\lambda}}_1^T \mathbf{x} \right\} dt$$

$$H_1 \triangleq \boldsymbol{\lambda}_1^T \mathbf{f}[\mathbf{x},\mathbf{u}]$$

Euler-Lagrange Equations and 1st Variation for Terminal Constraint "Cost" Stationarity

$$\boldsymbol{\lambda}_1(t_f) = \left\{ \frac{\partial \psi[\mathbf{x}(t_f)]}{\partial \mathbf{x}} \right\}^T$$

$$\dot{\boldsymbol{\lambda}}_1 = -\left\{ \frac{\partial H_1[\mathbf{x},\mathbf{u},\boldsymbol{\lambda}_1,t]}{\partial \mathbf{x}} \right\}^T = -\left[\boldsymbol{\lambda}_1^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]^T = -\left[\mathbf{F}^T \boldsymbol{\lambda}_1 \right]$$

Assuming that these equations are satisfied, the first variation is

$$\Delta J_1 = \int_{t_o}^{t_f} \left(\frac{\partial H_1}{\partial \mathbf{u}} \Delta \mathbf{u} \right) dt = \int_{t_o}^{t_f} \left(\boldsymbol{\lambda}_1^T \mathbf{G} \Delta \mathbf{u} \right) dt$$

First Variation of the Constrained Cost

$$\Delta J_C = \Delta J_0 + \mu \Delta J_1$$
$$= 0 \text{ for constrained stationarity}$$

$$\Delta J_0 = \int_{t_o}^{t_f} \left(\frac{\partial H_0}{\partial \mathbf{u}} \Delta \mathbf{u} \right) dt \qquad \Delta J_1 = \int_{t_o}^{t_f} \left(\frac{\partial H_1}{\partial \mathbf{u}} \Delta \mathbf{u} \right) dt$$

$$\Delta J_C = \Delta J_0 + \mu \Delta J_1$$
$$= \int_{t_o}^{t_f} \left(\frac{\partial H_0}{\partial \mathbf{u}} + \mu \frac{\partial H_1}{\partial \mathbf{u}} \right) \Delta \mathbf{u}\, dt = \int_{t_o}^{t_f} \left[\left(L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G} \right) + \mu \boldsymbol{\lambda}_1^T \mathbf{G} \right] \Delta \mathbf{u}\, dt$$

First Variation of the Constrained Cost

$$\begin{aligned}\Delta J_C &= \Delta J_0 + \mu \Delta J_1 = 0 \\ &= \int_{t_o}^{t_f} \left(\frac{\partial H_0}{\partial \mathbf{u}} + \mu \frac{\partial H_1}{\partial \mathbf{u}} \right) \Delta \mathbf{u} \, dt = \int_{t_o}^{t_f} \left[\left(L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G} \right) + \mu \boldsymbol{\lambda}_1^T \mathbf{G} \right] \Delta \mathbf{u} \, dt\end{aligned}$$

Control perturbation is arbitrary, so chose

$$\Delta \mathbf{u} = \varepsilon \left(\frac{\partial H_1}{\partial \mathbf{u}} \right)^T = \varepsilon \left(\boldsymbol{\lambda}_1^T \mathbf{G} \right)^T, \quad \varepsilon = \text{ arbitrary constant}$$

First Variation of the Constrained Cost

$$\Delta J_C = \int_{t_o}^{t_f} \left[\left(L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G} \right) + \mu \boldsymbol{\lambda}_1^T \mathbf{G} \right] \varepsilon \mathbf{G}^T \boldsymbol{\lambda}_1 \, dt$$

$$= \varepsilon \int_{t_o}^{t_f} \left[\left(L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G} \right) \mathbf{G}^T \boldsymbol{\lambda}_1 + \mu \boldsymbol{\lambda}_1^T \mathbf{G} \mathbf{G}^T \boldsymbol{\lambda}_1 \right] dt$$

$$= \varepsilon \left\{ \int_{t_o}^{t_f} \left[\left(L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G} \right) \mathbf{G}^T \boldsymbol{\lambda}_1 \right] dt + \mu \int_{t_o}^{t_f} \left[\boldsymbol{\lambda}_1^T \mathbf{G} \mathbf{G}^T \boldsymbol{\lambda}_1 \right] dt \right\} \triangleq \varepsilon (a + \mu b)$$

Solution for terminal constraint Lagrange multiplier

$$\Delta J_C = 0 \quad \text{if} \quad \mu = -\frac{a}{b}$$

Controllability Gramian

For control of the terminal constraint, the controllability gramian must not equal zero

$$b \triangleq \int_{t_o}^{t_f} \left[\boldsymbol{\lambda}_1^T \mathbf{G}\mathbf{G}^T \boldsymbol{\lambda}_1 \right] dt \neq 0$$

A sufficient condition for optimality

Optimizing Control for Terminal Constraint

Choose $\mathbf{u}(t)$ such that

$$\frac{\partial H_C}{\partial \mathbf{u}} = \left[\frac{\partial H_0}{\partial \mathbf{u}} - \left(\frac{a}{b} \right) \frac{\partial H_1}{\partial \mathbf{u}} \right]$$

$$= \left[\left(L_{\mathbf{u}} + \boldsymbol{\lambda}_0^T \mathbf{G} \right) - \left(\frac{a}{b} \right) \boldsymbol{\lambda}_1^T \mathbf{G} \right] = 0$$

Linear, Time-Invariant Minimum-Time Problem

Linear, Time-Invariant Minimum-Time Problem

Linear, time-invariant system, scalar control

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,u(t), \quad \mathbf{x}(0) = \mathbf{x}_o$$

Control constraint

$$c(u) = |u| - 1 \le 0$$

Cost function

$$J = \int_0^{t_f} dt$$

Terminal constraint

$$\psi\left[\mathbf{x}\left(t_f\right)\right] = 0$$

Linear, Time-Invariant Minimum-Time Problem

Hamiltonian

$$H_C = 1 + \boldsymbol{\lambda}^T \left(\mathbf{F}\mathbf{x} + \mathbf{G}u \right) + \mu\psi$$

Adjoint equation

$$\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial H_C}{\partial \mathbf{x}} \right)^T = -\mathbf{F}^T \boldsymbol{\lambda}, \qquad \boldsymbol{\lambda}(t_f) = \left[\frac{\partial \psi}{\partial \mathbf{x}}(t_f) \right]$$

Open-end time problem

$$H_C{}^*(t_f) = 0$$

Time-invariant problem

$$H_C{}^*(t) = 0 \text{ on entire trajectory}$$

Linear, Time-Invariant Minimum-Time Problem

Optimality conditions not satisfied

$$\frac{\partial H_C}{\partial u} = \boldsymbol{\lambda}^T \mathbf{G}, \quad \therefore \frac{\partial^2 H_C}{\partial u^2} = 0 \Rightarrow \text{Singular problem (not convex)}$$

Minimum principle (smallest Hamiltonian) solves the problem

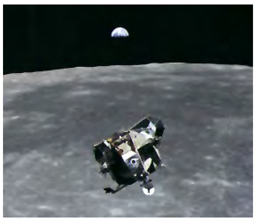
$$1 + \boldsymbol{\lambda}^{*T} \left(\mathbf{F}\mathbf{x}^* + \mathbf{G}u^* \right) \le 1 + \boldsymbol{\lambda}^{*T} \left(\mathbf{F}\mathbf{x}^* + \mathbf{G}u \right)$$

or

$$\boldsymbol{\lambda}^{*T} \left(\mathbf{G}u^* \right) \le \boldsymbol{\lambda}^{*T} \left(\mathbf{G}u \right), \quad \text{most negative value}$$

Optimal control is a switching law

$$u^* = \begin{cases} +1, & \boldsymbol{\lambda}^{*T}\,\mathbf{G} < 0 \\ -1, & \boldsymbol{\lambda}^{*T}\,\mathbf{G} > 0 \end{cases}$$



"Bang-Bang" Control of the Lunar Module

Second-order system with ON/OFF reaction control

$$\begin{bmatrix} \dot{\theta}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ g_A / I_{yy} \end{bmatrix} u(t)$$

Time evolution of the state while a thruster is on [$u(t) = 1$]

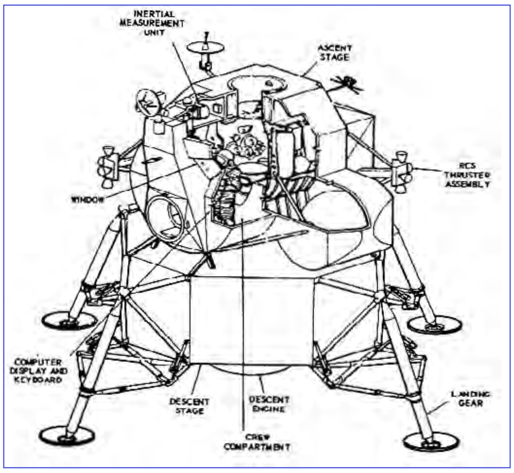
Angular rate, deg/s: $q(t) = \left(g_A / I_{yy}\right) t + q(0)$

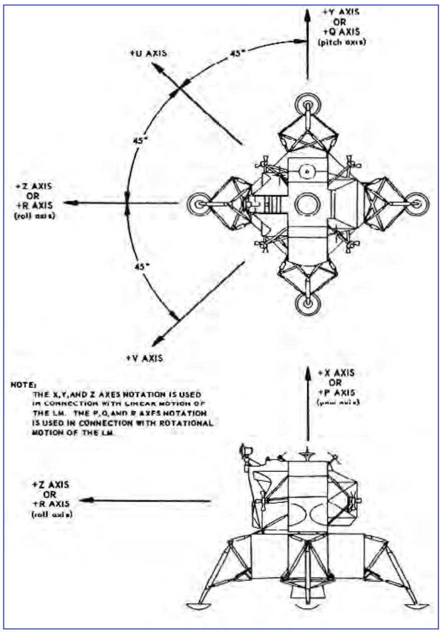
Angle, deg: $\theta(t) = \left(g_A / I_{yy}\right) t^2 / 2 + q(0)t + \theta(0)$

Neglecting initial conditions, what does the phase-plane plot (pitch rate vs. pitch angle) look like?

Apollo Lunar Module Control

- 16 reaction control thrusters
  - Control about 3 axes
  - Redundancy of thrusters
- LM Digital Autopilot



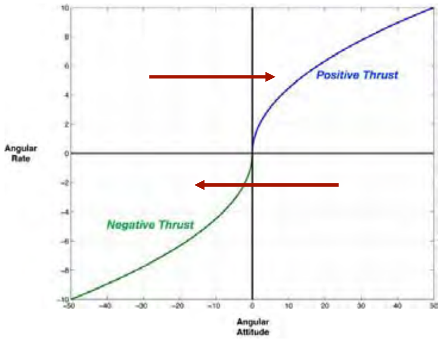


Constant-Thrust (Acceleration) Trajectories

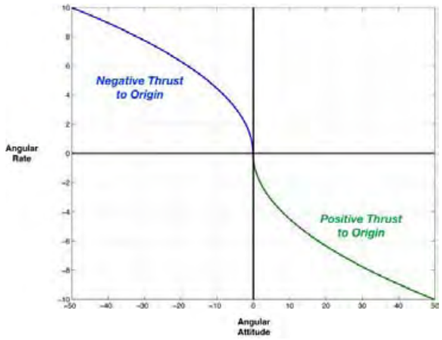
For $u = 1$,
Acceleration $= g_A/I_{yy}$

For $u = -1$,
Acceleration $= -g_A/I_{yy}$

Thrusting away from the origin

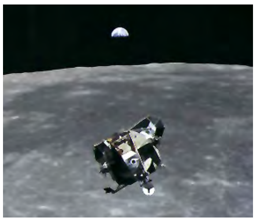


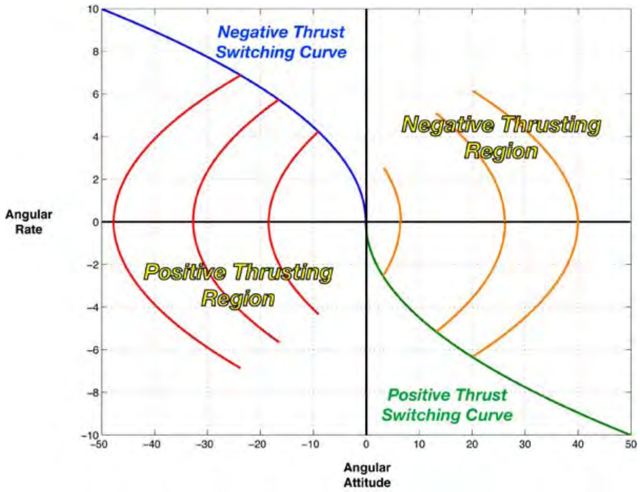
Thrusting to the origin



With zero thrust, what does the phase-plane plot look like?

Switching-Curve Control Law for On-Off Thrusters





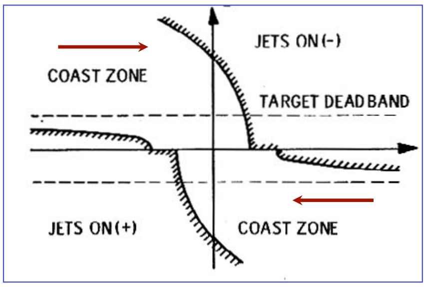
- Origin (i.e., zero rate and attitude error) can be reached from any point in the state space
- Control logic:
  - Thrust in one direction until switching curve is reached
  - Then reverse thrust
  - Switch thrust off when errors are zero

Next Time: Constraints and Numerical Optimization

Reading
OCE: Section 3.5, 3.6

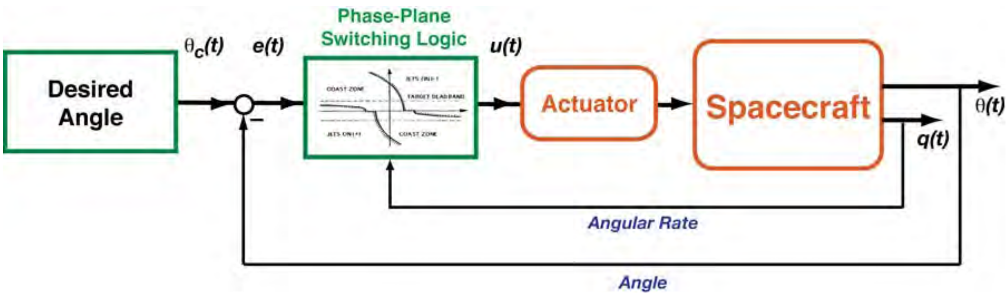
Supplementary Material

Apollo Lunar Module Phase-Plane Control Logic



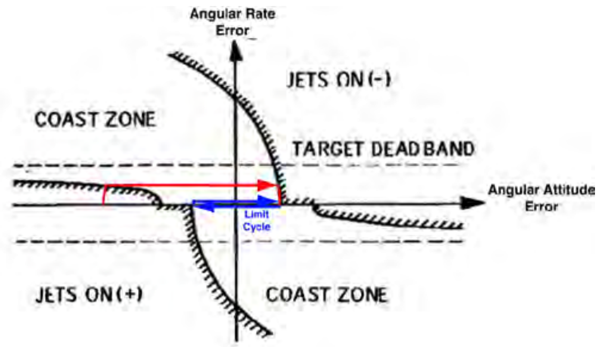
- Coast zones conserve RCS propellant by limiting angular rate
- With no coast zone, thrusters would chatter on and off at origin, wasting propellant
- State limit cycles about target attitude
- Switching curve shapes modified to provide robustness against modeling errors
  – RCS thrust level
  – Moment of inertia

Apollo Lunar Module Phase-Plane Control Law



- Switching logic implemented in the Apollo Guidance & Control Computer
- More efficient than a linear control law for on-off actuators

Typical Phase-Plane Trajectory



- With angle error, RCS turned on until reaching OFF switching curve
- Phase point drifts until reaching ON switching curve
- RCS turned off when rate is 0-
- Limit cycle maintained with minimum-impulse RCS firings
  - Amplitude = ±1 deg (coarse), ±0.1 deg (fine)