Path Constraints and Numerical Optimization

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- State and Control Equality Constraints
- Pseudoinverse
- State and Control Inequality Constraints
- Numerical Methods
  - Penalty function and collocation
  - Neighboring optimal
  - Quasilinearization
  - Gradient
    - Newton-Raphson
    - Steepest descent
- Optimal Treatment of an Infection

http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Minimization with Equality Constraints

Minimization with Equality Constraint on State and Control

$$\min_{\mathbf{u}(t)} J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right] dt$$

- **subject to**

Dynamic Constraint

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)], \quad \mathbf{x}\left(t_o\right) \text{ given}$$

$\dim(\mathbf{x}) = n \times 1$
$\dim(\mathbf{f}) = n \times 1$
$\dim(\mathbf{u}) = m \times 1$

State/Control Equality Constraint

$$\mathbf{c}\left[\mathbf{x}(t),\mathbf{u}(t)\right] \equiv 0 \text{ in } (t_o, t_f); \quad \dim(\mathbf{c}) = \left(r \times 1\right) \le \left(m \times 1\right)$$

Aeronautical Example:

Longitudinal Point-Mass Dynamics

$$\dot{V} = \left(T - C_D \frac{1}{2}\rho V^2 S\right)\Big/ m - g\sin\gamma$$

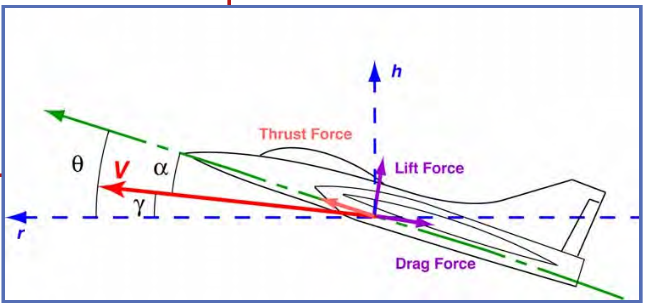
$$\dot{\gamma} = \frac{1}{V}\left[\left(C_L \frac{1}{2}\rho V^2 S\right)\Big/ m - g\cos\gamma\right]$$

$$\dot{h} = V\sin\gamma$$

$$\dot{r} = V\cos\gamma$$

$$\dot{m} = -(SFC)(T)$$

$x_1 = V$: Velocity, m/s
$x_2 = \gamma$: Flight path angle, rad
$x_3 = h$: Height, m
$x_4 = r$: Range, m
$x_5 = m$: Mass, kg



Aeronautical Example:
Longitudinal Point-Mass Dynamics

$u_1 = \delta T$: Throttle setting, %
$u_2 = \alpha$: Angle of attack, rad

$T = \mathrm{T}_{\max_{SL}} \left(e^{-\beta h}\right) \delta T$: Thrust, N
$C_D = \left(C_{D_o} + \varepsilon C_L{}^2\right)$ Drag coefficient
$C_L = C_{L_\alpha} \alpha =$ Lift coefficient
$S =$ Reference area, m^2
$m =$ Vehicle mass, kg
$\rho =$ Air density $= \rho_{SL} e^{-\beta h}$, $\mathrm{kg/m}^3$
$g =$ Gravitational acceleration, $\mathrm{m/s}^2$
SFC = Specific Fuel Consumption, g/kN-s

Path Constraint Included in the Cost Function Hamiltonian

Constraint must be satisfied at every instant of the trajectory
Dimension of the constraint ≤ dimension of the control

$$J_1 = \psi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} \left\{L + \boldsymbol{\lambda}_1^T(t)\left[\mathbf{f} - \dot{\mathbf{x}}(t)\right] + \boldsymbol{\mu}^T \mathbf{c}\right\} dt$$

$$\mathbf{c}\left[\mathbf{x}(t), \mathbf{u}(t)\right] \equiv 0 \quad \text{in } (t_o, t_f)$$

The constraint is adjoined to the Hamiltonian

$$H \triangleq L + \boldsymbol{\lambda}_1^T \mathbf{f} + \boldsymbol{\mu}^T \mathbf{c}$$

$\dim(\mathbf{x}) = \dim(\mathbf{f}) = \dim(\boldsymbol{\lambda}) = n \times 1$
$\dim(\mathbf{u}) = m \times 1$
$\dim(\mathbf{c}) = \dim(\boldsymbol{\mu}) = r \times 1, r \le m$

Euler-Lagrange Equations Including Equality Constraint

$$\boldsymbol{\lambda}(t_f) = \left\{ \frac{\partial \phi[\mathbf{x}(t_f)]}{\partial \mathbf{x}} \right\}^T$$

$$\dot{\boldsymbol{\lambda}} = -\left\{ \frac{\partial H[\mathbf{x},\mathbf{u},\boldsymbol{\lambda},\mathbf{c},\boldsymbol{\mu},t]}{\partial \mathbf{x}} \right\}^T = -\left[\frac{\partial L}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \boldsymbol{\mu}^T \frac{\partial \mathbf{c}}{\partial \mathbf{x}} \right]^T = -\left[L_{\mathbf{x}}^T + \mathbf{F}^T \boldsymbol{\lambda} + \left(\frac{\partial \mathbf{c}}{\partial \mathbf{x}} \right)^T \boldsymbol{\mu} \right]$$

$$\left\{ \frac{\partial H[\mathbf{x},\mathbf{u},\boldsymbol{\lambda},\mathbf{c},\boldsymbol{\mu},t]}{\partial \mathbf{u}} \right\}^T = -\left[\left(\frac{\partial L}{\partial \mathbf{u}} \right)^T + \mathbf{G}^T \boldsymbol{\lambda} + \left(\frac{\partial \mathbf{c}}{\partial \mathbf{u}} \right)^T \boldsymbol{\mu} \right] = \mathbf{0}$$

$$\mathbf{c}\left[\mathbf{x}(t),\mathbf{u}(t)\right] \equiv 0 \ \text{ in } (t_o, t_f)$$

$$\begin{aligned}
&\dim(\mathbf{x}) = \dim(\boldsymbol{\lambda}) = n \times 1 \\
&\dim(\mathbf{F}) = n \times n \\
&\dim(\mathbf{G}) = n \times m \\
&\dim(\mathbf{u}) = m \times 1 \\
&\dim(\mathbf{c}) = \dim(\boldsymbol{\mu}) = r \times 1, r \le m
\end{aligned}$$

No Optimization When *r* = *m*

- **Control entirely specified by constraint**
  - ***m* unknowns, *m* equations**

$$\mathbf{c}\left[\mathbf{x}(t),\mathbf{u}(t)\right] \equiv \mathbf{0} \Rightarrow \mathbf{u}(t) = fcn\left[\mathbf{x}(t)\right]$$

Example

$$\mathbf{c} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) = \mathbf{0}; \quad \dim(\mathbf{x}) = n \times 1; \quad \dim(\mathbf{c}) = \dim(\mathbf{u}) = m \times 1$$

$$\dim(\mathbf{A}) = m \times n; \quad \dim(\mathbf{B}) = m \times m$$

$$\mathbf{u}(t) = -\mathbf{B}^{-1}\mathbf{A}\mathbf{x}(t)$$

- **Constraint Lagrange multiplier is irrelevant but can be expressed**
  - **from *dH*/*du* = 0,**

$$\boldsymbol{\mu} = -\left(\frac{\partial \mathbf{c}}{\partial \mathbf{u}} \right)^{-T} \left[\left(\frac{\partial L}{\partial \mathbf{u}} \right)^T + \mathbf{G}^T \boldsymbol{\lambda} \right]$$

$$\left(\frac{\partial \mathbf{c}}{\partial \mathbf{u}} \right) \text{ is square and non-singular}$$

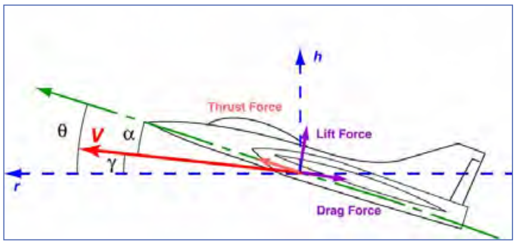
No Optimization When $r = m$

MAINTAIN CONSTANT VELOCITY AND FLIGHT PATH ANGLE

$$\mathbf{c}\left[\mathbf{x}(t),\mathbf{u}(t)\right] \equiv \mathbf{0} = \begin{bmatrix} 0 = \dot{V} = \left[T_{\max}\delta T - \left(C_{D_o} + \varepsilon C_L^2\right)\frac{1}{2}\rho V^2 S\right]\Big/ m - g\sin\gamma \\ 0 = \dot{\gamma} = \frac{1}{V}\left[\left(C_{L_\alpha}\alpha\frac{1}{2}\rho V^2 S\right)\Big/ m - g\cos\gamma\right] \end{bmatrix}$$

$$\Rightarrow \mathbf{u}(t) = fcn\left[\mathbf{x}(t)\right]$$

$$V(0) = V_{desired}$$
$$\gamma(0) = \gamma_{desired}$$



Effect of Constraint Dimensionality: $r < m$

MINIMIZE FUEL AND CONTROL USE WHILE MAINTAINING CONSTANT FLIGHT PATH ANGLE

$$\min_{\mathbf{u}(t)} J = \int_{t_o}^{t_f}\left(q\dot{m}^2 + r_1 u_1^2 + r_2 u_2^2\right)dt$$

$$c\left[\mathbf{x}(t),\mathbf{u}(t)\right] \equiv 0 = \dot{\gamma} = \frac{1}{V(t)}\left[\frac{\left(C_{L_\alpha}\alpha\frac{1}{2}\rho V^2(t) S\right)}{m} - g\cos\gamma(t)\right]$$

$$\gamma(0) = \gamma_{desired}$$

Effect of Constraint Dimensionality: *r* < *m*

$$\dim(\mathbf{x}) = n \times 1$$
$$\dim(\mathbf{u}) = m \times 1$$
$$\dim(\mathbf{c}) = r \times 1$$

$$\left(\frac{\partial \mathbf{c}}{\partial \mathbf{u}}\right) \text{ is not square when } r < m$$

$$\therefore \left(\frac{\partial \mathbf{c}}{\partial \mathbf{u}}\right) \text{ is not strictly invertible}$$

- **Three approaches to constrained optimization**
  - Algebraic solution for *r* control variables using an invertible subset of the constraint
  - Pseudoinverse of control effect
  - "Soft" constraint

Effect of Constraint Dimensionality: *r* < *m*

$$\dim(\mathbf{u}) = m \times 1$$
$$\dim(\mathbf{c}) = r \times 1$$

Algebraic solution for *r* control variables using an invertible subset of the constraint

Example 1

$$\dim(\mathbf{x}) = n \times 1; \quad \dim(\mathbf{A}_r) = r \times n$$

$$\dim(\mathbf{u}) = m \times 1; \quad \dim(\mathbf{u}_r) = r \times 1; \quad \dim(\mathbf{B}_r) = r \times r$$

$$\mathbf{c} = \mathbf{A}_r \mathbf{x}(t) + \mathbf{B}_r \mathbf{u}_r(t) = \mathbf{0}; \quad \det(\mathbf{B}_r) \neq 0$$

$$\mathbf{u}_r(t) = -\mathbf{B}_r^{-1}\mathbf{A}\mathbf{x}(t)$$

Example 2

$$\dim(\mathbf{u}) = m \times 1; \quad \dim(\mathbf{u}_r) = r \times 1; \quad \dim(\mathbf{B}_1) = r \times r$$

$$\mathbf{c} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_r \\ \mathbf{u}_{m-r} \end{bmatrix} = \mathbf{B}_1 \mathbf{u}_r + \mathbf{B}_2 \mathbf{u}_{m-r} = \mathbf{0}; \quad \det(\mathbf{B}_1) \neq 0$$

$$\mathbf{u}_r(t) = -\mathbf{B}_1^{-1}\mathbf{B}_2 \mathbf{u}_{m-r}(t)$$

Second Approach: Satisfy Constraint Using Left Pseudoinverse: $r < m$

$$\left[\left(\frac{\partial \mathbf{c}}{\partial \mathbf{u}}\right)^T\right]^L$$ is the **left pseudoinverse** of control sensitivity

$$\dim\left[\left(\frac{\partial \mathbf{c}}{\partial \mathbf{u}}\right)^T\right]^L = r \times m$$

Lagrange multiplier

$$\boldsymbol{\mu}_L = -\left[\left(\frac{\partial \mathbf{c}}{\partial \mathbf{u}}\right)^T\right]^L \left[\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T + \mathbf{G}^T \boldsymbol{\lambda}\right]$$

Pseudoinverse of Matrix

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

$$\dim(\mathbf{x}) = r \times 1$$
$$\dim(\mathbf{y}) = m \times 1$$

$$(m \times 1) = (m \times r)(r \times 1)$$

$r = m$, **A** is square and non-singular

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$$

$$(r \times 1) = (r \times m)(m \times 1) = (r \times r)(r \times 1)$$

$r \neq m$, **A** is not square

Use pseudoninverse of **A**

$$\mathbf{x} = \mathbf{A}^{\#}\mathbf{y} = \mathbf{A}^{\dagger}\mathbf{y}$$

$$(r \times 1) = (r \times m)(m \times 1)$$

Maximum rank of **A** is r or m, whichever is smaller

See http://en.wikipedia.org/wiki/Moore-Penrose_pseudoinverse

Left Pseudoinverse

$\dim(\mathbf{x}) = r \times 1$
$\dim(\mathbf{y}) = m \times 1$

Maximum rank of A is *r* or *m*, whichever is smaller

$$\dim\left(\mathbf{A}^T\mathbf{A}\right) = r \times r$$
$$\dim\left(\mathbf{A}\mathbf{A}^T\right) = m \times m$$

***r* < *m*, Left pseudoinverse is appropriate**

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$
$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{y}$$

Averaging solution

$$\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{y}$$
$$\mathbf{A}^L \triangleq \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$$
$$\mathbf{x} = \mathbf{A}^L\mathbf{y}$$

$(r \times 1) = (r \times m)(m \times 1)$

Right Pseudoinverse

$\dim(\mathbf{x}) = r \times 1$
$\dim(\mathbf{y}) = m \times 1$

***r* > *m*, Right pseudoinverse is appropriate**

$\dim\left(\mathbf{A}^T\mathbf{A}\right) = r \times r$
$\dim\left(\mathbf{A}\mathbf{A}^T\right) = m \times m$

$$\mathbf{A}\mathbf{x} = \mathbf{y} = \mathbf{I}\mathbf{y}$$
$$\mathbf{A}\mathbf{x} = \left(\mathbf{A}\mathbf{A}^T\right)\left(\mathbf{A}\mathbf{A}^T\right)^{-1}\mathbf{y}$$
$$= \mathbf{I}\mathbf{y}$$

Minimum Euclidean error norm solution

$$\mathbf{A}\mathbf{x} = \mathbf{A}\left[\mathbf{A}^T\left(\mathbf{A}\mathbf{A}^T\right)^{-1}\mathbf{y}\right]$$
$$\mathbf{x} = \mathbf{A}^T\left(\mathbf{A}\mathbf{A}^T\right)^{-1}\mathbf{y}$$
$$\mathbf{A}^R \triangleq \mathbf{A}^T\left(\mathbf{A}\mathbf{A}^T\right)^{-1}$$
$$\mathbf{x} = \mathbf{A}^R\mathbf{y}$$

$(m \times 1) = (m \times r)(r \times 1)$

Left Pseudoinverse Example

$$\mathbf{Ax} = \mathbf{y}, \quad r < m$$
$$\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{y}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} x = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$x = \left\{\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}^{-1}\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix}$$
$$= \frac{1}{10}(20) = 2$$

Unique solution

Right Pseudoinverse Example

$$\mathbf{Ax} = \mathbf{y}, \quad r > m$$
$$\mathbf{x} = \mathbf{A}^T\left(\mathbf{AA}^T\right)^{-1}\mathbf{y}$$

$$\mathbf{Ax} = \mathbf{y}; \quad \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 14$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\left\{\begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}^{-1} 14$$
$$= \frac{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}{10}(14) = \begin{bmatrix} 1.4 \\ 4.2 \end{bmatrix}$$

At least two solutions

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ satisfies the equation, but $\|\mathbf{x}\|_2 = \sqrt{20}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 4.2 \end{bmatrix}$ and $\|\mathbf{x}\|_2 = \sqrt{19.6}$

Minimum - norm solution

Necessary Conditions Use Left Pseudoinverse for *r* < *m*

Optimality conditions

$$\boldsymbol{\lambda}(t_f) = \left\{\frac{\partial\phi[\mathbf{x}(t_f)]}{\partial\mathbf{x}}\right\}^T \qquad \dot{\boldsymbol{\lambda}} = -\left[L_{\mathbf{x}}^T + \mathbf{F}^T\boldsymbol{\lambda} + \left(\frac{\partial\mathbf{c}}{\partial\mathbf{x}}\right)^T\boldsymbol{\mu}\right]$$

$$\left[\left(\frac{\partial L}{\partial\mathbf{u}}\right)^T + \mathbf{G}^T\boldsymbol{\lambda} + \left(\frac{\partial\mathbf{c}}{\partial\mathbf{u}}\right)^T\boldsymbol{\mu}\right] = 0$$

with

$$\mathbf{c}\left[\mathbf{x}(t),\mathbf{u}(t)\right] \equiv 0$$

Third Approach: Penalty Function Provides "Soft" State-Control Equality Constraint: *r* < *m*

$$L \triangleq L_{original} + \varepsilon\mathbf{c}^T\mathbf{c}$$

ε : Scalar penalty weight

Euler-Lagrange equations are adjusted accordingly

$$\boldsymbol{\lambda}(t_f) = \left\{\frac{\partial\phi[\mathbf{x}(t_f)]}{\partial\mathbf{x}}\right\}^T$$

$$\dot{\boldsymbol{\lambda}} = -\left[L_{\mathbf{x}}^T + \mathbf{F}^T\boldsymbol{\lambda}\right] = -\left[\left(\frac{\partial L_{orig}}{\partial\mathbf{x}} + 2\varepsilon\mathbf{c}^T\frac{\partial\mathbf{c}}{\partial\mathbf{x}}\right)^T + \mathbf{F}^T\boldsymbol{\lambda}\right]$$

$$\left[\left(\frac{\partial L_{orig}}{\partial\mathbf{u}} + 2\varepsilon\mathbf{c}^T\frac{\partial\mathbf{c}}{\partial\mathbf{u}}\right)^T + \mathbf{G}^T\boldsymbol{\lambda}\right] = 0$$

$$\mathbf{c}\left[\mathbf{x}(t),\mathbf{u}(t)\right] \equiv 0$$

Equality Constraint on State Alone

$$\boxed{\mathbf{c}\left[\mathbf{x}(t)\right] \equiv 0 \ \text{ in } (t_o, t_f)}$$

$$J_1 = \psi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} \left\{ L + \boldsymbol{\lambda}_1^T(t)\left[\mathbf{f} - \dot{\mathbf{x}}(t)\right] + \boldsymbol{\mu}^T \mathbf{c} \right\} dt$$

Hamiltonian

$$H \triangleq L + \boldsymbol{\lambda}_1^T \mathbf{f} + \boldsymbol{\mu}^T \mathbf{c}$$

Constraint is insensitive to control perturbations to first order

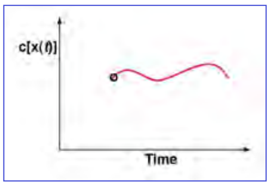
$$\Delta\mathbf{c} = \left(\frac{\partial \mathbf{c}}{\partial \mathbf{x}}\right)\Delta\mathbf{x} + \underbrace{\cancel{\left(\frac{\partial \mathbf{c}}{\partial \mathbf{u}}\right)}}_{0}\Delta\mathbf{u} = \left(\frac{\partial \mathbf{c}}{\partial \mathbf{x}}\right)\Delta\mathbf{x}$$

Example of Equality Constraint on State Alone

MINIMIZE FUEL AND CONTROL USE WHILE MAINTAINING CONSTANT ALTITUDE

$$\min_{\mathbf{u}(t)} J = \int_{t_o}^{t_f} \left(q\dot{m}^2 + r_1 u_1^2 + r_2 u_2^2 \right) dt$$

$$c\left[\mathbf{x}(t), \mathbf{u}(t)\right] = c\left[\mathbf{x}(t)\right] = 0 = h(t) - h_{desired}$$



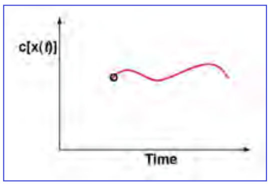
Introduce Time-Derivative of Equality Constraint

Equality constraint has no effect on optimality condition

$$\left\{\frac{\partial H}{\partial \mathbf{u}}\right\}^T = -\left[\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T + \mathbf{G}^T\boldsymbol{\lambda} + \left(\cancel{\frac{\partial \mathbf{c}}{\partial \mathbf{u}}}\right)^T \boldsymbol{\mu}\right]$$
$$= -\left[\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T + \mathbf{G}^T\boldsymbol{\lambda}\right]$$

Solution: Incorporate time-derivative of c[x(t)] in optimization

Introduce Time-Derivative of Equality Constraint



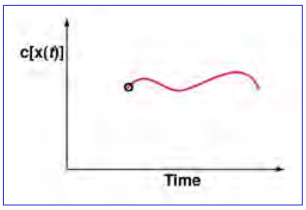
Define c[x(t)] as the zeroth-order equality constraint

$$\mathbf{c}\left[\mathbf{x}(t)\right] \triangleq \mathbf{c}^{(0)}\left[\mathbf{x}(t)\right] \equiv 0$$

Compute first-order equality constraint

$$\frac{d\mathbf{c}^{(0)}\left[\mathbf{x}(t)\right]}{dt} = \frac{\partial \mathbf{c}^{(0)}\left[\mathbf{x}(t)\right]}{\partial t} + \frac{\partial \mathbf{c}^{(0)}\left[\mathbf{x}(t)\right]}{\partial \mathbf{x}} \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right]$$
$$\triangleq \mathbf{c}^{(1)}\left[\mathbf{x}(t),\mathbf{u}(t)\right] = 0$$

Time-Derivative of Equality Constraint



Optimality condition now includes derivative of equality constraint

$$\left\{\frac{\partial H}{\partial \mathbf{u}}\right\}^T = -\left[\left(\frac{\partial L}{\partial \mathbf{u}}\right)^T + \mathbf{G}^T\boldsymbol{\lambda} + \left(\frac{\partial \mathbf{c}^{(1)}}{\partial \mathbf{u}}\right)^T \boldsymbol{\mu}\right]$$

Subject to

$$\mathbf{c}^{(0)}\left[\mathbf{x}(t_o)\right] \equiv 0 \ \underline{\text{or}} \ \mathbf{c}^{(0)}\left[\mathbf{x}(t_f)\right] \equiv 0$$

- With equality constraint satisfied at beginning or end of trajectory, $\mathbf{c}^{(1)} = 0$ assures that constraint is satisfied throughout
- If $\partial \mathbf{c}^{(1)}/\partial \mathbf{u} = 0$, differentiate again, and again, …

State Equality Constraint Example

$$c\left[\mathbf{x}(t)\right] \triangleq c^{(0)}\left[\mathbf{x}(t)\right] = 0 = h(t) - h_{desired}$$

No control in the constraint; differentiate

$$\frac{d\mathbf{c}^{(0)}\left[\mathbf{x}(t)\right]}{dt} = \frac{\partial \mathbf{c}^{(0)}\left[\mathbf{x}(t)\right]}{\partial t} + \frac{\partial \mathbf{c}^{(0)}\left[\mathbf{x}(t)\right]}{\partial \mathbf{x}}\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right]$$
$$\triangleq \mathbf{c}^{(1)}\left[\mathbf{x}(t)\right] = 0 = \dot{h}(t) = V(t)\sin\gamma(t)$$

Still no control in the constraint; differentiate again…

State Equality Constraint Example

Still no control in the constraint; differentiate again

$$\frac{d\mathbf{c}^{(1)}[\mathbf{x}(t)]}{dt} = \frac{\partial \mathbf{c}^{(1)}[\mathbf{x}(t)]}{\partial t} + \frac{\partial \mathbf{c}^{(1)}[\mathbf{x}(t)]}{\partial \mathbf{x}} \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)]$$

$$\triangleq \mathbf{c}^{(2)}[\mathbf{x}(t)] = 0 = \ddot{h}(t) = \frac{dV}{dt}(t)\sin\gamma(t) + V(t)\frac{d[\sin\gamma(t)]}{dt}$$

$$= \left[\left(T_{\max_{SL}}\left(e^{-\beta h}\right)\delta T - C_D \frac{1}{2}\rho V^2 S\right)\Big/ m - g\sin\gamma\right]\sin\gamma(t)$$

$$+\cos\gamma(t)\left[\left(C_{L_\alpha}\alpha\frac{1}{2}\rho V^2 S\right)\Big/ m - g\cos\gamma\right]$$

State Equality Constraint Example

- Control appears in the 2nd-order equality constraint

$$\mathbf{c}^{(2)}[\mathbf{x}(t), \mathbf{u}(t)] = 0$$

$$= \left[\left(T_{\max_{SL}}\left(e^{-\beta h}\right)\delta T - C_D(\alpha)\frac{1}{2}\rho(h)V^2(t)S\right)\Big/ m(t) - g\sin\gamma(t)\right]\sin\gamma(t)$$

$$+\cos\gamma(t)\left[\left(C_{L_\alpha}\alpha\frac{1}{2}\rho(h)V^2(t)S\right)\Big/ m(t) - g\cos\gamma(t)\right]$$

$$\boxed{H \triangleq L + \boldsymbol{\lambda}_1^T \mathbf{f} + \boldsymbol{\mu}^T \mathbf{c}^{(2)}}$$

- 0th- and 1st-order constraints satisfied at some point on the trajectory (e.g., t_0)

$$c^{(0)}[\mathbf{x}(t_0)] = 0 \quad \Rightarrow h(t_0) = h_{desired}$$

$$c^{(1)}[\mathbf{x}(t_0)] = 0 \quad \Rightarrow \gamma(t_0) = 0$$

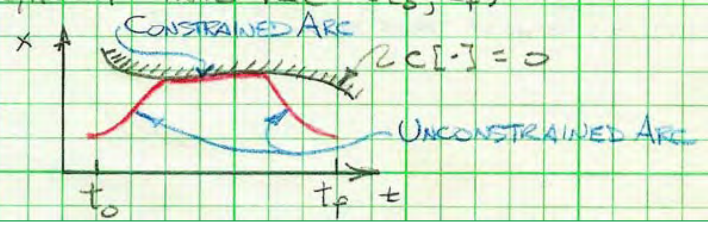
Minimization with Inequality Constraints

"Hard" Inequality Constraints

Inequality Constraints on State and Control in (t_o, t_f)

- Principal Difference from Equality Constraint:

  Inequality constraint normally not invoked throughout interval (t_o, t_f):

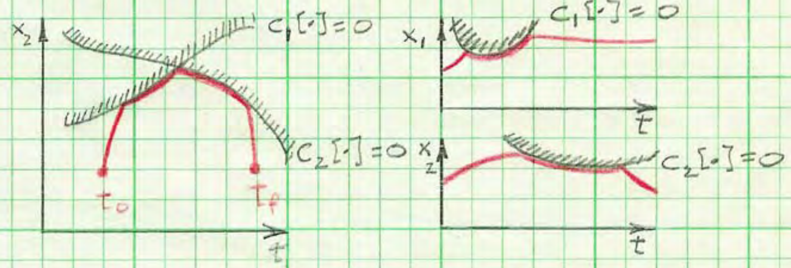


Inequality Constraints

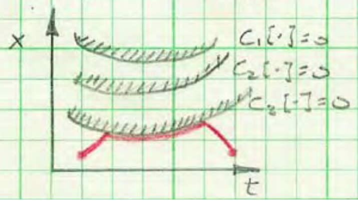
- DIFFERENT ELEMENTS OF

  $\underline{c}[\underline{x}(t), \underline{u}(t), t] \le 0$

  MAY BE INVOKED AT DIFFERENT TIMES:



- $\dim(\underline{c}) = r \gtrsim m, n$. CONSTRAINTS CAN BLOCK EACH OTHER.

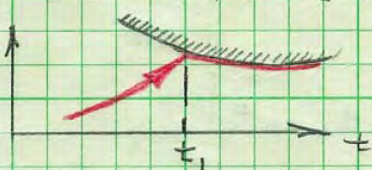


Inequality Constraints

WEIERSTRASS - ERDMANN CONDITIONS

- $c[\underline{x}(t), \underline{u}(t), t]$ OR $c[\underline{u}(t), t] \le 0$

  - JUNCTURE BETWEEN UNCONSTRAINED & CONSTRAINED "ARCS" [ENTRY POINT] @ $t = t_1$:



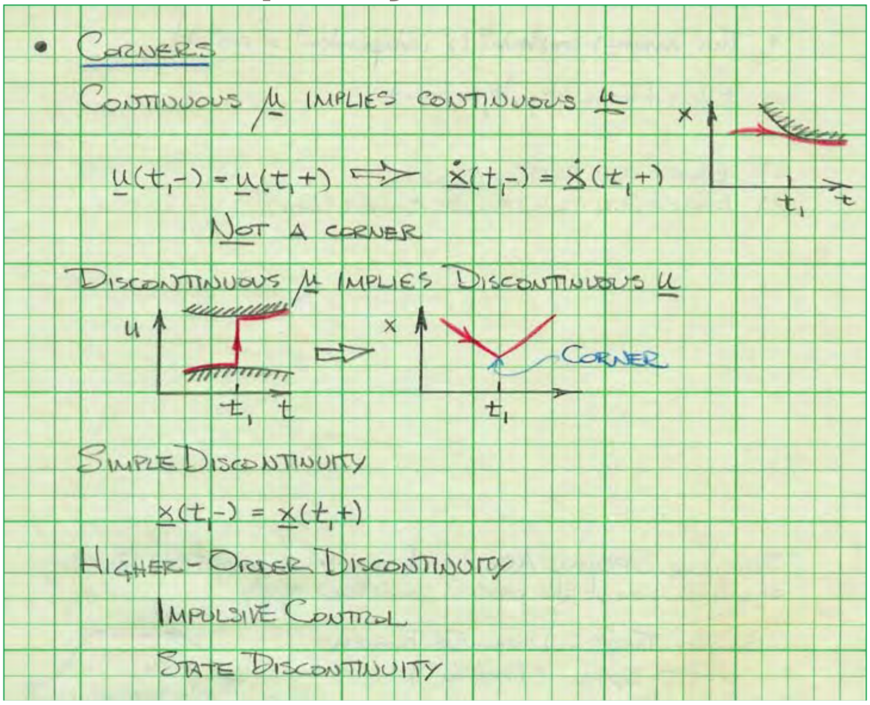
  - 3 TRANSVERSALITY CONDITIONS

$$\underline{\lambda}(t_1-) = \underline{\lambda}(t_1+)$$

$$\mathcal{H}(t_1-) = \mathcal{H}(t_1+)$$

$$\mathcal{H}_u(t_1-) = \mathcal{H}_u(t_1+) \quad \gtrless 0$$

Inequality Constraints



- CORNERS

CONTINUOUS $\underline{\mu}$ IMPLIES CONTINUOUS $\underline{u}$

$\underline{u}(t_1-) = \underline{u}(t_1+) \Rightarrow \dot{\underline{x}}(t_1-) = \dot{\underline{x}}(t_1+)$

NOT A CORNER

DISCONTINUOUS $\underline{\mu}$ IMPLIES DISCONTINUOUS $\underline{u}$

CORNER

SIMPLE DISCONTINUITY

$\underline{x}(t_1-) = \underline{x}(t_1+)$

HIGHER-ORDER DISCONTINUITY

IMPULSIVE CONTROL

STATE DISCONTINUITY

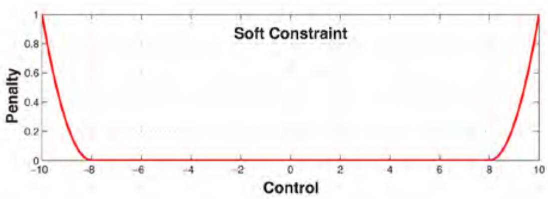
"Soft" Control Inequality Constraint

$$L \triangleq L_{original} + \varepsilon \mathbf{c}^T \mathbf{c}$$

ε : Scalar penalty weight

Scalar Example

$$c[u(t)] = \begin{cases} (u - u_{\max})^2 & , \quad u \geq u_{\max} \\ 0 & , \quad u_{\min} < u < u_{\max} \\ (u - u_{\min})^2 & , \quad u \leq u_{\min} \end{cases}$$



Numerical Optimization

Numerical Optimization Methods

	Optimality of Solution	Solution Method			Iteration Variables	Order of ODE[a] Solution
		$\mathbf{x}(t)$	$\boldsymbol{\lambda}(t)$	$\mathbf{u}(t)$		
Parametric	approximate	ODE[a]	–	I^b	$\mathbf{u}(\mathbf{k}_u, t)$	n
Penalty function	approximate	I	–	I	$\mathbf{x}(\mathbf{k}_x, t), \mathbf{u}(\mathbf{k}_u, t)$	none
Dynamic programming	exact	ODE	PDE[c]	I	$\mathbf{u}(t)$	n
Neighboring extremal	exact	ODE	ODE	$\mathscr{H}_\mathbf{u} = \mathbf{0}$	$\boldsymbol{\lambda}(t_0)$	$2n$
Quasilinearization	exact	I	I	$\mathscr{H}_\mathbf{u} = \mathbf{0}$	$\mathbf{x}(t), \boldsymbol{\lambda}(t)$	$2n^d$
Gradient	exact	ODE	ODE	I	$\mathbf{u}(t)$	$2n$

[a] ODE: ordinary differential equation.

[b] Iteration.

[c] PDE: Partial differential equation; HJB equation; one dependent variable (V), $(n+1)$ independent variables $(\mathbf{x}, t)$, $\partial V/\partial \mathbf{x}$ corresponds to $\boldsymbol{\lambda}^T$.

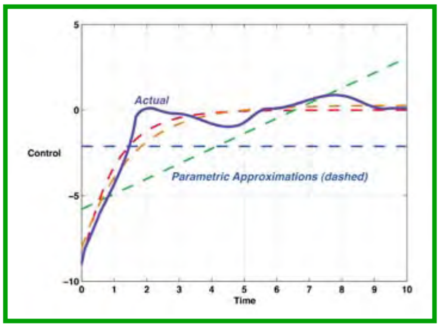
[d] Perturbation equation for $\Delta\mathbf{x}(t)$ and $\Delta\boldsymbol{\lambda}(t)$.

Parametric Optimization

$$\min_{\mathbf{u}(t)} J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right]dt$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)], \quad \mathbf{x}(t_o)\ given$$



Control specified by a parameter vector, k

No adjoint equations

Examples

$$u(t) = k$$

$$u(t) = k_0 + k_1 t + k_2 t^2 + \cdots, \quad \mathbf{k} = \begin{bmatrix} k_0 & k_1 & \cdots & k_m \end{bmatrix}^T$$

$$\mathbf{u}(t) = \mathbf{k}$$

$$\mathbf{u}(t) = \mathbf{k}_0 + \mathbf{k}_1 t + \mathbf{k}_2 t^2 + \cdots, \quad \mathbf{k} = \begin{bmatrix} \mathbf{k}_0 & \mathbf{k}_1 & \cdots & \mathbf{k}_m \end{bmatrix}$$

$$\mathbf{u}(t) = \mathbf{k}_0 + \mathbf{k}_1 \sin\left(\frac{\pi t}{t_f - t_0}\right) + \mathbf{k}_2 \cos\left(\frac{\pi t}{t_f - t_0}\right), \quad \mathbf{k} = \begin{bmatrix} \mathbf{k}_0 & \mathbf{k}_1 & \mathbf{k}_2 \end{bmatrix}$$

Parametric Optimization

$$\min_{\mathbf{u}(t)} J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right]dt$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),\mathbf{u}(t)], \quad \mathbf{x}(t_o)\ given$$

- **Necessary and sufficient conditions for a minimum**
- **Use static search algorithm to find minimizing control parameter, k**

$$\frac{\partial J}{\partial \mathbf{k}} = 0$$

$$\frac{\partial^2 J}{\partial \mathbf{k}^2} > 0$$

Parametric Optimization Example

$$\min_{u(t)} J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right]dt$$

subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t),u(t)]\,, \quad \mathbf{x}(t_o)\ given$$

$$u(t) \triangleq \left(k_0 + k_1 t + k_2 t^2\right)$$

$$\dot{V} = \left(T_{\max} - C_D \frac{1}{2}\rho V^2 S\right)\Big/ m - g\sin\gamma$$

$$\dot{\gamma} = \frac{1}{V}\left\{\left[C_L\left(k_0 + k_1 t + k_2 t^2\right)\frac{1}{2}\rho V^2 S\right]\Big/ m - g\cos\gamma\right\}$$

$$\dot{h} = V\sin\gamma$$

$$\dot{r} = V\cos\gamma$$

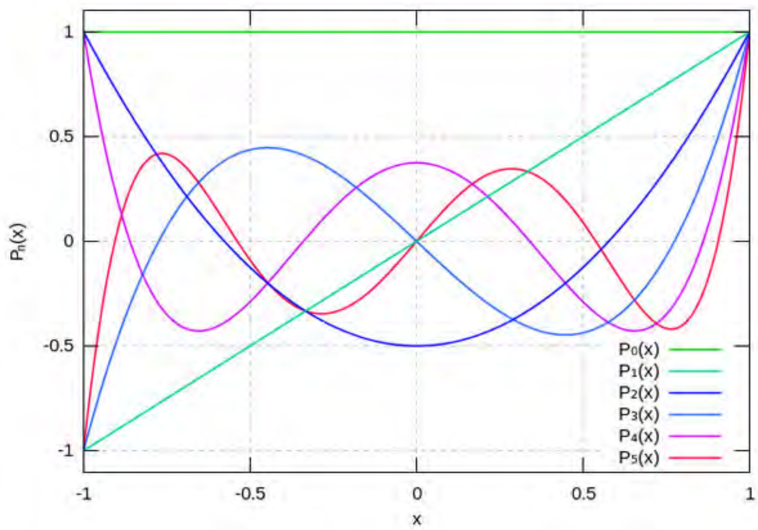
$$\dot{m} = -(SFC)(T)$$

$$\mathbf{a} = \begin{bmatrix} k_0 \\ k_1 \\ k_2 \end{bmatrix}$$

$$\frac{\partial J}{\partial \mathbf{k}} = 0$$

$$\frac{\partial^2 J}{\partial \mathbf{k}^2} > 0$$

Legendre Polynomials



Solutions to Legendre's differential equation

Legendre Polynomials

Polynomials can be generated by Rodriques's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]$$

Optimizing Control: Find minimizing values of k_n

$$u(x) = k_0 P_0(x) + k_1 P_1(x) + k_2 P_2(x) + k_3 P_3(x) + k_4 P_4(x) + k_5 P_5(x) + \cdots$$

$$x \triangleq \frac{t}{t_f - t_o}$$

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$
$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$
$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$
$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

Control History Optimized with Legendre Polynomials Could be expressed as a Simple Power Series

$$u^*(x) = k_0^* P_0(x) + k_1^* P_1(x) + k_2^* P_2(x) + k_3^* P_3(x) + k_4^* P_4(x) + k_5^* P_5(x) + \cdots$$

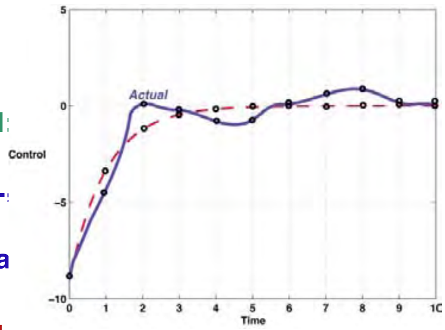
$$u^*(x) = a_0^* + a_1^* x + a_2^* x^2 + a_3^* x^3 + a_4^* x^4 + a_5^* x^5 + \cdots$$

$$a_0^* = k_0^* - k_2^*\left(\frac{1}{2}\right) + k_4^*\left(\frac{3}{8}\right) + \cdots$$
$$a_1^* = k_1^* - k_3^*\left(\frac{3}{2}\right) + k_5^*\left(\frac{15}{8}\right) + \cdots$$
$$a_2^* = k_2^*\left(\frac{3}{2}\right) - k_4^*\left(\frac{30}{8}\right) + \cdots$$
$$\cdots$$

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$
$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$
$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$
$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

Parametric Optimization: Collocation

- Admissible controls occur at discrete times, k
- Cost and dynamic constraint are discretized
- "Pseudospectral" Optimal Control: State and adjoint points may be connected by basis functions, e.g., Legendre polynomials
- Continuous solution approached a time interval decreased



$$\min_{\mathbf{u}_k} J = \phi\left[\mathbf{x}_{k_f}\right] + \sum_{k=0}^{k_f-1} L\left[\mathbf{x}_k, \mathbf{u}_k\right]$$

subject to

$$\mathbf{x}_{k+1} = \mathbf{f}_k[\mathbf{x}_k, \mathbf{u}_k], \quad \mathbf{x}_0 \ given$$

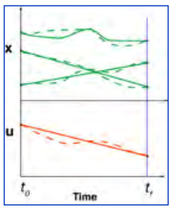
http://en.wikipedia.org/wiki/Collocation_method

http://en.wikipedia.org/wiki/Legendre_polynomials

http://en.wikipedia.org/wiki/Pseudospectral_optimal_control

Penalty Function Method

Balakrishnan's "Epsilon" Technique



- No integration of the dynamic equation
- Parametric optimization of the state and control history

$$\begin{aligned}\mathbf{x}(t) &\equiv \mathbf{x}(\mathbf{k}_\mathbf{x}, t)\\ \mathbf{u}(t) &\equiv \mathbf{u}(\mathbf{k}_\mathbf{u}, t)\end{aligned} \qquad \begin{aligned}\dim(\mathbf{k}_\mathbf{x}) &\geq n\\ \dim(\mathbf{k}_\mathbf{u}) &\geq m\end{aligned}$$

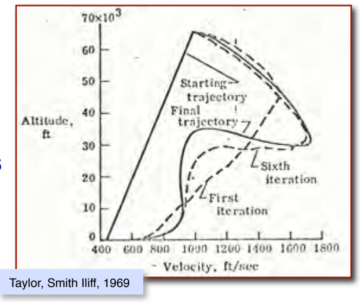
- Augment the integral cost function by the dynamic equation error

$$\min_{wrt\ \mathbf{u}(t), \mathbf{x}(t)} J = \varphi\left[\mathbf{x}(t_f), t_f\right] + \int_{t_o}^{t_f} \left\{ L\left[\mathbf{x}(t), \mathbf{u}(t), t\right] + \left(\frac{1}{\varepsilon}\right) \left(\left\{ \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), t\right] - \dot{\mathbf{x}}(t) \right\}^T \{\bullet\} \right) \right\} dt$$

$1/\varepsilon$ is the penalty for not satisfying the dynamic constraint

Penalty Function Method

- **Choose reasonable starting values of state and control parameters**
  - **e.g., state and control satisfy boundary conditions**
- **Evaluate cost function**



Taylor, Smith Iliff, 1969

$$J_0 = \varphi\left[\mathbf{x}_0(t_f)\right] + \int_{t_o}^{t_f} \left\{ L\left[\mathbf{x}_0(t), \mathbf{u}\ (t)\right] + \left(\frac{1}{\varepsilon}\right)\left(\left\{\mathbf{f}\left[\mathbf{x}_0(t), \mathbf{u}_0(t)\right] - \dot{\mathbf{x}}_0(t)\right\}^T \{\bullet\}\right)\right\} dt$$

Update state and control parameters (e.g., steepest descent)

$$\mathbf{k}_{\mathbf{x}_{i+1}} = \mathbf{k}_{\mathbf{x}_i} - \alpha \left[\left.\frac{\partial J}{\partial \mathbf{k}_{\mathbf{x}}}\right|_{\mathbf{k}_{\mathbf{x}} = \mathbf{k}_{\mathbf{x}_i}} \right]^T \qquad \mathbf{k}_{\mathbf{u}_{i+1}} = \mathbf{k}_{\mathbf{u}_i} - \alpha \left[\left.\frac{\partial J}{\partial \mathbf{k}_{\mathbf{u}}}\right|_{\mathbf{k}_{\mathbf{u}} = \mathbf{k}_{\mathbf{u}_i}} \right]^T$$

$$\begin{aligned} \mathbf{x}_{i+1}(t) &\equiv \mathbf{x}(\mathbf{k}_{\mathbf{x}_{i+1}}, t) \\ \mathbf{u}_{i+1}(t) &\equiv \mathbf{u}(\mathbf{k}_{\mathbf{u}_{i+1}}, t) \end{aligned}$$

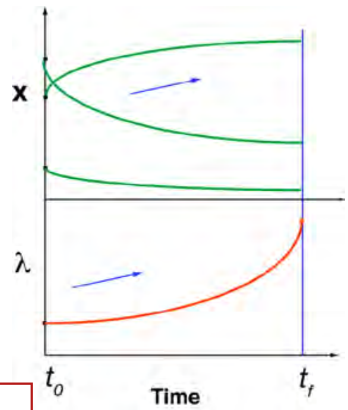
Re-evaluate cost with higher penalty

Repeat to convergence

$$J_i \rightarrow J_{i+1} \rightarrow J^*, \quad \varepsilon \rightarrow 0, \quad \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), t\right] \rightarrow \dot{\mathbf{x}}(t)$$

Neighboring Extremal Method

"Shooting Method": Integrate both state and adjoint vector forward in time



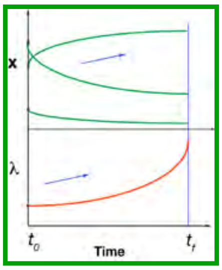
$$\begin{aligned} &\dot{\mathbf{x}}_{k+1}(t) = \mathbf{f}[\mathbf{x}_{k+1}(t), \mathbf{u}_k(t)], \\ &\quad \mathbf{x}_0(t_0) \text{ given, initial guess for } \mathbf{u}_0(t) \end{aligned}$$

$$\dot{\boldsymbol{\lambda}}_{k+1}(t) = -\left[\left.\frac{\partial L}{\partial \mathbf{x}}\right|_k (t) + \boldsymbol{\lambda}_{k+1}^T(t) \mathbf{F}_k(t) \right]^T, \quad \boldsymbol{\lambda}_k(t_0) \ \text{ given}$$

with

$$\mathbf{u}_{k+1}(t) \text{ defined by } \frac{\partial H[\mathbf{x}_k(t), \mathbf{u}_k(t), \boldsymbol{\lambda}_{k+1}(t), t]}{\partial \mathbf{u}} = \left[L_{\mathbf{u}_k}(t) + \boldsymbol{\lambda}_{k+1}^T(t) \mathbf{G}_k(t) \right] = \mathbf{0}$$

... but how do you know the initial value of the adjoint vector?



Neighboring Extremal Method

All trajectories are optimal (i.e., "extremals") for some cost function because

$$\frac{\partial H}{\partial \mathbf{u}} = H_{\mathbf{u}} = \left[L_{\mathbf{u}} + \boldsymbol{\lambda}^T \mathbf{G} \right] = \mathbf{0}$$

Integrating state equation computes a value for $\phi\left[\mathbf{x}\left(t_f\right)\right]$

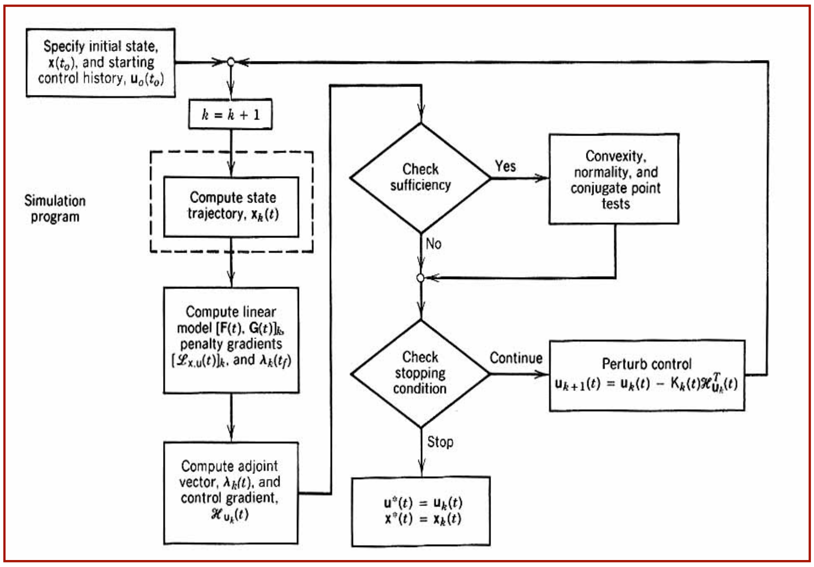
$$\mathbf{x}(t_f) = \mathbf{x}(t_0) + \int_{t_0}^{t_f} \mathbf{f}[\mathbf{x}_{k+1}(t), \mathbf{u}_k(t)]; \quad \phi\left[\mathbf{x}(t_f)\right] \rightarrow \frac{\partial \phi\left[\mathbf{x}(t_f)\right]}{\partial \mathbf{x}} = \boldsymbol{\lambda}^{\mathrm{T}}\left(t_f\right)$$

Use a learning rule to estimate the initial value of the adjoint vector, e.g.,

$$\boldsymbol{\lambda}_{k+1}^T \left(t_0\right) = \boldsymbol{\lambda}_k^T\left(t_0\right) - \alpha\left[\boldsymbol{\lambda}_k^T\left(t_f\right) - \boldsymbol{\lambda}_{desired}\right]^T$$

Gradient-Based Methods

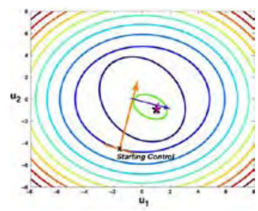
Gradient-Based Search Algorithms



Gradient-Based Search Algorithms

Steepest Descent

$$\mathbf{u}_{k+1}(t) = \mathbf{u}_k(t) - \varepsilon_k \left[\frac{\partial H}{\partial \mathbf{u}}(t)\right]_k^T$$



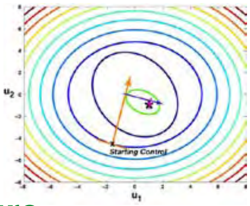
Newton Raphson

$$\mathbf{u}_{k+1}(t) = \mathbf{u}_k(t) - \left[\frac{\partial^2 H}{\partial \mathbf{u}^2}(t)\right]_k^{-1} \left[\frac{\partial H}{\partial \mathbf{u}}(t)\right]_k^T$$

Generalized Direct Search

$$\mathbf{u}_{k+1}(t) = \mathbf{u}_k(t) - \mathbf{K}_k \left[\frac{\partial H}{\partial \mathbf{u}}(t)\right]_k^T$$

Numerical Optimization Using Steepest-Descent Algorithm



Iterative bidirectional procedure

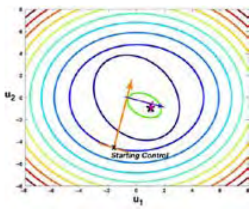
Forward solution to find the state, $\mathbf{x}(t)$

Backward solution to find the adjoint vector, $\boldsymbol{\lambda}(t)$

Steepest-descent adjustment of control history, $\mathbf{u}(t)$

$$\dot{\mathbf{x}}_k(t) = \mathbf{f}[\mathbf{x}_k(t), \mathbf{u}_{k-1}(t)], \quad \mathbf{x}(t_o) \textit{ given}$$

Use educated guess for $\mathrm{u}_0(t)$ on first iteration

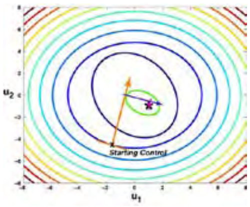


Numerical Optimization Using Steepest-Descent Algorithm

$$\dot{\boldsymbol{\lambda}}_k(t) = -\left[\frac{\partial H}{\partial \mathbf{x}}\right]_k^T = -\left[L_{\mathbf{x}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{F}(t)\right]_k^T,$$

$$\boldsymbol{\lambda}(t_f) = \left\{\frac{\partial \phi[\mathbf{x}(t_f)]}{\partial \mathbf{x}}\right\}^T \qquad [E - L \ \#2 \textit{ and } \#1]$$

Use $x_{k-1}(t)$ and $u_{k-1}(t)$ from previous step



Numerical Optimization Using Steepest-Descent Algorithm

$$\left(\frac{\partial H}{\partial \mathbf{u}}\right)_k = \left[L_{\mathbf{u}}(t) + \boldsymbol{\lambda}^T(t)\mathbf{G}(t)\right]_k \qquad \left[\boldsymbol{E-L\ \#3}\right]$$

$$\mathbf{u}_{k+1}(t) = \mathbf{u}_k(t) - \varepsilon\left[\left.\frac{\partial H}{\partial \mathbf{u}}\right|_{\mathbf{u}(t)=\mathbf{u}_k(t)}\right]^T$$
$$= \mathbf{u}_k(t) - \varepsilon\left[L_{\mathbf{u}} + \boldsymbol{\lambda}^T(t)\mathbf{G}(t)\right]_k^T$$

Use $\mathbf{x}(t)$, $\boldsymbol{\lambda}(t)$, and $\mathbf{u}(t)$ from previous step

Finding the Best Steepest-Descent Gain

$J_{0_k}\left[u_k(t),\quad 0<t<t_f\right]$: **Best solution from the previous iteration**

Calculate the gradient, $\frac{\partial H_k}{\partial u}(t)$, in $0<t<t_f$

$J_{1_k}\left[u_k(t)-\varepsilon_k\frac{\partial H_k}{\partial u}(t),\quad 0<t<t_f\right]$: **Steepest - descent calculation of cost (1)**

$J_{2_k}\left[u_k(t)-2\varepsilon_k\frac{\partial H_k}{\partial u}(t),\quad 0<t<t_f\right]$: **Steepest - descent calculation of cost (2)**

$$J(\varepsilon) = a_0 + a_1\varepsilon + a_2\varepsilon^2$$

$$\begin{bmatrix} J_{0_k} \\ J_{1_k} \\ J_{2_k} \end{bmatrix} = \begin{bmatrix} a_0 + a_1(0) + a_2(0)^2 \\ a_0 + a_1(\varepsilon_k) + a_2(\varepsilon_k)^2 \\ a_0 + a_1(2\varepsilon_k) + a_2(2\varepsilon_k)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & (\varepsilon_k) & (\varepsilon_k)^2 \\ 1 & (2\varepsilon_k) & (2\varepsilon_k) \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Solve for a_0, a_1, and a_2

Find ε^* **that minimizes** $J(\varepsilon)$

$J_{k+1}\left[u_{k+1}(t) = u_k(t) - \varepsilon^*_k\frac{\partial H_k}{\partial u}(t),\quad 0<t<t_f\right]$: **Best steepest - descent calculation of cost**

Go to next iteration

Steepest-Descent Algorithm for Problem with Terminal Constraint

$$\min_{\mathbf{u}(t)} J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f} L\left[\mathbf{x}(t),\mathbf{u}(t)\right] dt$$

$$\psi\left[\mathbf{x}(t_f)\right] \equiv 0 \quad \text{(scalar)}$$

$$\frac{\partial H_C}{\partial \mathbf{u}} = \left[\frac{\partial H_0}{\partial \mathbf{u}} - \left(\frac{a}{b}\right)\frac{\partial H_1}{\partial \mathbf{u}}\right] = 0$$

see Lecture 3 for a and b definitions

Chose $\mathbf{u}_{k+1}(t)$ such that

$$\mathbf{u}_{k+1}(t) = \mathbf{u}_k(t) - \varepsilon\left[\left.\frac{\partial H_C}{\partial \mathbf{u}}\right|_{\mathbf{u}(t)=\mathbf{u}_k(t)}\right]^T$$

$$= \mathbf{u}_k(t) - \varepsilon\left[L_{\mathbf{u}}{}^T + \mathbf{G}^T(t)\boldsymbol{\lambda}_0(t)\right]_k - \frac{1}{b_k}\mathbf{G}^T{}_k(t)\boldsymbol{\lambda}_1(t)\psi_k\left[\mathbf{x}(t_f)\right]$$

Zero Gradient Algorithm for Quadratic Control Cost

$$\min_{\mathbf{u}(t)} J = \phi\left[\mathbf{x}(t_f)\right] + \int_{t_o}^{t_f}\left\{L\left[\mathbf{x}(t)\right] + \frac{1}{2}\mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right\} dt$$

$$H\left[\mathbf{x}(t),\mathbf{u}(t),\boldsymbol{\lambda}(t)\right] = \left\{L\left[\mathbf{x}(t)\right] + \frac{1}{2}\mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\right\} + \boldsymbol{\lambda}^T(t)\mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t)\right]$$

Optimality condition:

$$\frac{\partial H}{\partial \mathbf{u}}(t) = H_{\mathbf{u}}(t) = \left[\mathbf{u}^T(t)\mathbf{R} + \boldsymbol{\lambda}^T(t)\mathbf{G}(t)\right] \equiv \mathbf{0}$$

Zero Gradient Algorithm for Quadratic Control Cost

$$\boxed{\frac{\partial H}{\partial \mathbf{u}}(t) = H_{\mathbf{u}}(t) = \left[\mathbf{u}^T(t)\mathbf{R} + \boldsymbol{\lambda}^T(t)\mathbf{G}(t)\right] \equiv \mathbf{0}}$$

Optimal control, u*(*t*)

$$\boxed{\begin{aligned}\mathbf{u}^{*T}(t)\mathbf{R} &= -\boldsymbol{\lambda}^{*T}(t)\mathbf{G}^*(t)\\ \mathbf{u}^*(t) &= -\mathbf{R}^{-1}\mathbf{G}^{*T}(t)\boldsymbol{\lambda}^*(t)\end{aligned}}$$

But $\mathbf{G}_k(t)$ and $\boldsymbol{\lambda}_k(t)$ are sub-optimal before convergence, and optimal control cannot be computed in single step

∴ Chose $\mathbf{u}_{k+1}(t)$ such that

$$\boxed{\begin{gathered}\mathbf{u}_{k+1}(t) = (1-\varepsilon)\mathbf{u}_k(t) - \varepsilon\left[\mathbf{R}^{-1}\mathbf{G}_k^T(t)\boldsymbol{\lambda}_k(t)\right]\\ \varepsilon \triangleq \text{Relaxation parameter} < 1\end{gathered}}$$

Stopping Conditions for Numerical Optimization

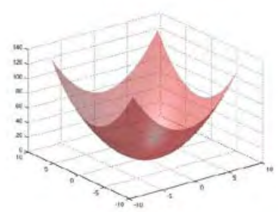
- **Computed total cost, *J*, reaches a theoretical minimum, e.g., zero** $\qquad \boxed{J_{k+1} = 0 + \varepsilon}$
- **Convergence of *J* is essentially complete** $\qquad \boxed{J_{k+1} > J_k - \varepsilon}$
- **Control gradient, $H_u(t)$, is essentially zero throughout $[t_o, t_f]$**

$$\boxed{\left|H_{\mathbf{u}_{k+1}}(t)\right| = 0 \pm \varepsilon \text{ in } \left[t_o, t_f\right] \quad \text{or} \quad \int_{t_o}^{t_f} H_{\mathbf{u}_{k+1}}^T(t) H_{\mathbf{u}_{k+1}}(t)\,dt = 0 + \varepsilon}$$

- **Terminal cost/constraint is satisfied, and integral cost is "good enough"**

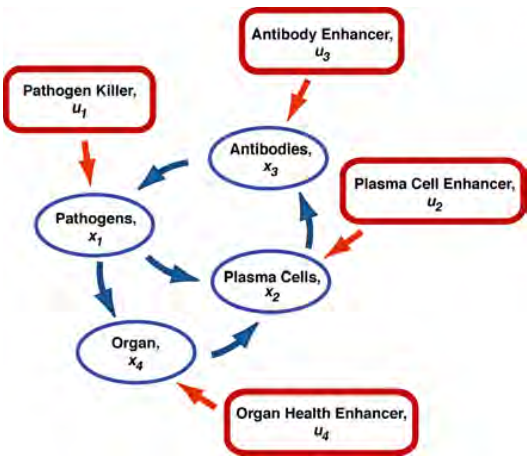
$$\boxed{\varphi_{k+1}\left(t_f\right) = 0 + \varepsilon, \text{ or } \psi_{k+1}\left(t_f\right) = 0 \pm \varepsilon, \text{ and } \int_{t_o}^{t_f} L\left[\mathbf{x}(t), \mathbf{u}(t)\right]dt < \delta}$$

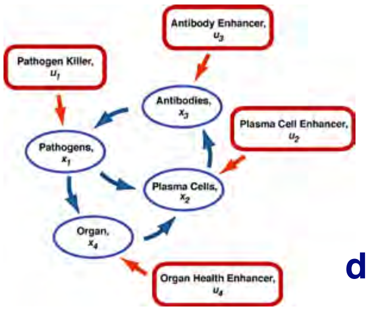
Optimal Treatment of an Infection



Model of Infection and Immune Response

- x_1 = Concentration of a **pathogen**, which displays antigen
- x_2 = Concentration of **plasma cells**, which are carriers and producers of antibodies
- x_3 = Concentration of **antibodies**, which recognize antigen and kill pathogen
- x_4 = Relative characteristic of a **damaged organ** [0 = healthy, 1 = dead]



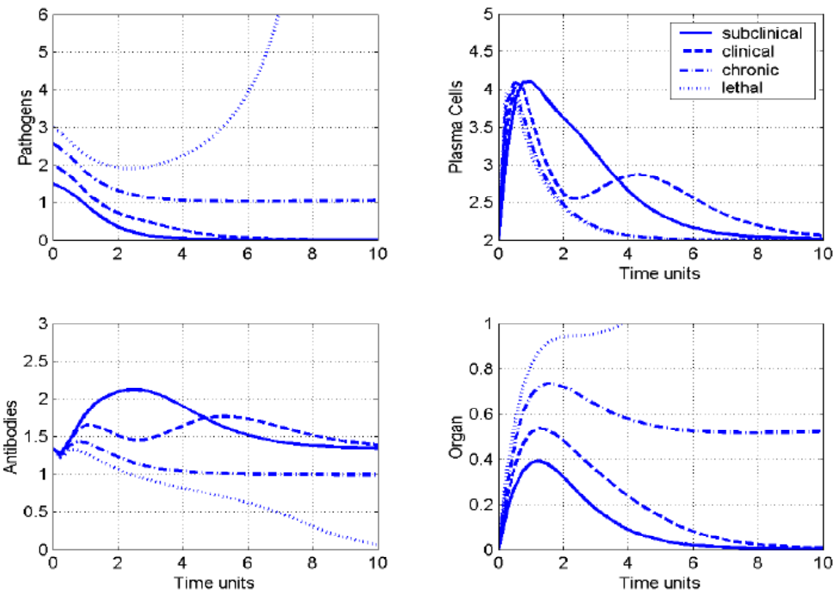


Infection Dynamics

Fourth-order ordinary differential equation, including effects of therapy (control)

$$\dot{x}_1 = (a_{11} - a_{12}x_3)x_1 + b_1u_1 + w_1$$
$$\dot{x}_2 = a_{21}(x_4)a_{22}x_1x_3 - a_{23}(x_2 - x_2{*}) + b_2u_2 + w_2$$
$$\dot{x}_3 = a_{31}x_2 - (a_{32} + a_{33}x_1)x_3 + b_3u_3 + w_3$$
$$\dot{x}_4 = a_{41}x_1 - a_{42}x_4 + b_4u_4 + w_4$$

Uncontrolled Response to Infection



Cost Function to be Minimized by Optimal Therapy

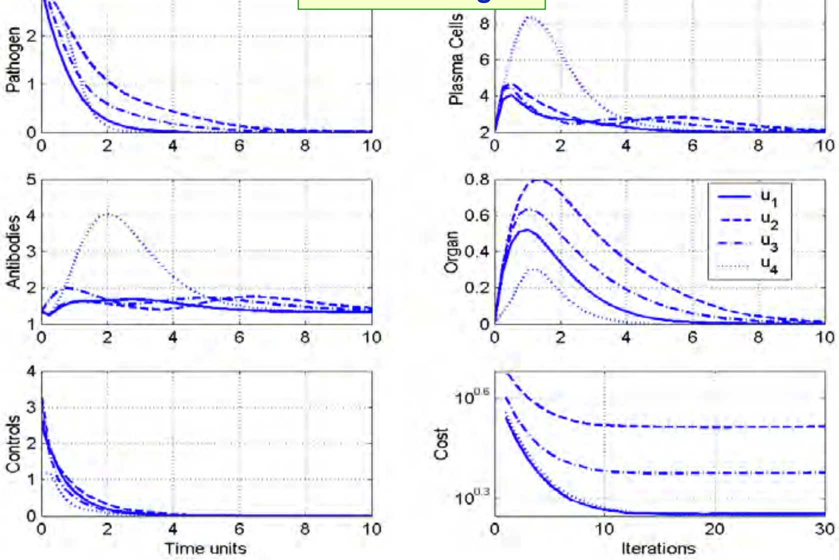
$$J = \frac{1}{2}\left(p_{11}x_{1_f}^2 + p_{44}x_{4_f}^2\right) + \frac{1}{2}\int_{t_o}^{t_f}\left(q_{11}x_1^2 + q_{44}x_4^2 + ru^2\right)dt$$

- Tradeoffs between final values, integral values over a fixed time interval, state, and control
- Cost function includes weighted square values of
  - Final concentration of the pathogen
  - Final health of the damaged organ (0 is good, 1 is bad)
  - Integral of pathogen concentration
  - Integral health of the damaged organ (0 is good, 1 is bad)
  - Integral of drug usage
- Drug cost may reflect physiological cost (side effects) or financial cost

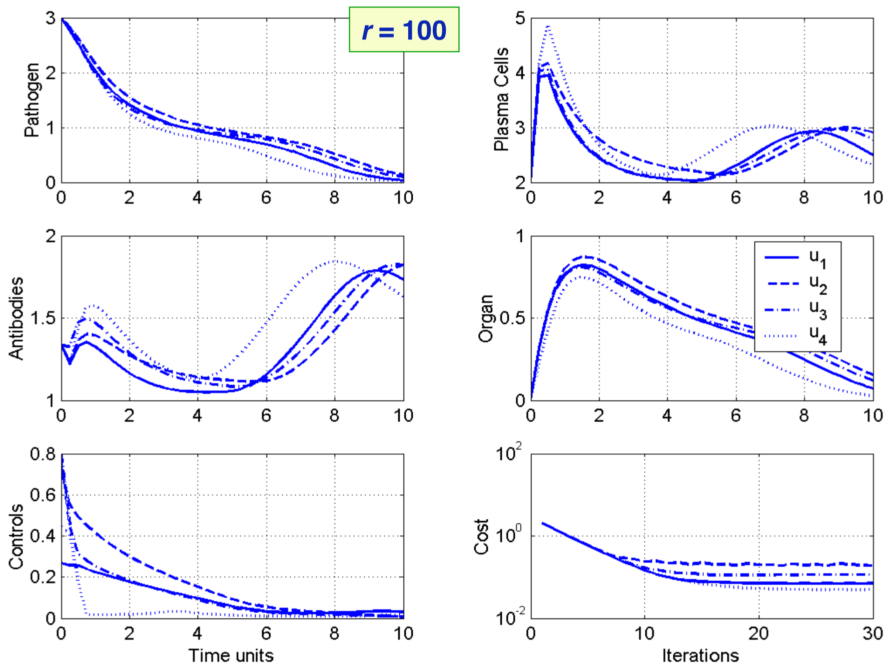
Examples of Optimal Therapy

- u_1 = Pathogen killer
- u_2 = Plasma cell enhancer
- u_3 = Antibody enhancer
- u_4 = Organ health enhancer

Unit cost weights



Effects of Increased Drug "Cost"

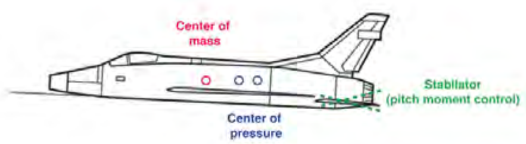


Next Time:
Minimum-Time and -Fuel Problems

Reading
OCE: Section 3.5, 3.6

Supplemental Material

Examples of Equality Constraints



$\mathbf{c}\left[\mathbf{x}(t),\mathbf{u}(t)\right] \equiv 0$

Pitch Moment = 0 = fcn(Mach Number, Stabilator Trim Angle)

$\mathbf{c}\left[\mathbf{u}(t)\right] \equiv 0$

Stabilator Trim Angle – constant = 0

$\mathbf{c}\left[\mathbf{x}(t)\right] \equiv 0$

Altitude – constant = 0

Minimum-Error-Norm Solution

$$\begin{aligned}&\dim(\mathbf{x}) = r \times 1\\&\dim(\mathbf{y}) = m \times 1\\&r > m\end{aligned}$$

- Euclidean error norm for linear equation

$$\boxed{\|\mathbf{Ax}-\mathbf{y}\|_2^2 = [\mathbf{Ax}-\mathbf{y}]^T[\mathbf{Ax}-\mathbf{y}]}$$

- Necessary condition for minimum error

$$\boxed{\frac{\partial}{\partial \mathbf{x}}\|\mathbf{Ax}-\mathbf{y}\|_2^2 = 2[\mathbf{Ax}-\mathbf{y}]^T = 0}$$

- Express **x** as right pseudoinverse

$$\begin{aligned}&2[\mathbf{Ax}-\mathbf{y}]^T = 2\left\{\mathbf{A}\left[\mathbf{A}^T\left(\mathbf{AA}^T\right)^{-1}\mathbf{y}\right]-\mathbf{y}\right\}^T = 2\left\{\left(\mathbf{AA}^T\right)\left(\mathbf{AA}^T\right)^{-1}\mathbf{y}-\mathbf{y}\right\}^T\\&= 2[\mathbf{y}-\mathbf{y}]^T = 0\end{aligned}$$

- Therefore, **x** is the minimizing solution, as long as $\mathbf{AA}^T$ is non-singular