Linearization of Nonlinear Models

- Most chemical process models are nonlinear, but they are often linearized to perform a simulation and stability analysis.

- Linear models are easier to understand (than nonlinear models) and are necessary for most control system design methods.

Single Variable Example

- A general single variable nonlinear model

$$\frac{dx}{dt} = f(x)$$

- The function *f*(*x*) can be approximated by a Taylor series approximation around the steady-state operating point (x_s)

$$f(x) = f(x_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s} (x - x_s) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_s} (x - x_s)^2 + \text{high order terms}$$

- Neglect the quadratic and higher order terms

$$f(x) \approx f(x_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s} (x - x_s)$$

At steady-state

$$\frac{dx_s}{dt} = f(x_s) = 0$$

The partial derivative of f(x) with respect to x, evaluated at the steady-state

$$\frac{dx}{dt} = f(x) \approx \left.\frac{\partial f}{\partial x}\right|_{x_s} (x - x_s)$$

- Since the derivative of a constant (x_s) is zero

$$\frac{dx}{dt} = \frac{d\left(x - x_s\right)}{dt}$$

$$\frac{d\left(x - x_s\right)}{dt} \approx \left.\frac{\partial f}{\partial x}\right|_{x_s} \left(x - x_s\right)$$

- We are often interested in deviations in a state from a steady-state operating point (**deviation variable**)

$$\frac{d\overline{x}}{dt} \approx \left.\frac{\partial f}{\partial x}\right|_{x_s} \overline{x}$$

$\overline{x} = x - x_s$: the change or perturbation from a steady-state value

- Write in state-space form

$$\frac{d\overline{x}}{dt} \approx a\,\overline{x} \qquad \text{where} \quad a = \left.\frac{\partial f}{\partial x}\right|_{x_s}$$

One State Variable and One Input Variable

- Consider a function with one state variable and one input variable

$$\dot{x} = \frac{dx}{dt} = f(x,u)$$

- Using a Taylor Series Expansion for $f(x,u)$

$$\dot{x} = f(x_s,u_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s,u_s}(x - x_s) + \left.\frac{\partial f}{\partial u}\right|_{x_s,u_s}(u - u_s) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_s,u_s}(x - x_s)^2$$

$$+ \left.\frac{\partial^2 f}{\partial x \partial u}\right|_{x_s,u_s}(x - x_s)(u - u_s) + \frac{1}{2}\left.\frac{\partial^2 f}{\partial u^2}\right|_{x_s,u_s}(u - u_s)^2 + \text{high order terms}$$

- Truncating after the linear terms

$$\dot{x} \approx f(x_s,u_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s,u_s}(x - x_s) + \left.\frac{\partial f}{\partial u}\right|_{x_s,u_s}(u - u_s)$$

zero

$$\frac{d(x - x_s)}{dt} \approx \left.\frac{\partial f}{\partial x}\right|_{x_s,u_s}(x - x_s) + \left.\frac{\partial f}{\partial u}\right|_{x_s,u_s}(u - u_s)$$

- Using deviation variables, $\bar{x} = x - x_s$ and $\bar{u} = u - u_s$

$$\frac{d\bar{x}}{dt} \approx \left.\frac{\partial f}{\partial x}\right|_{x_s, u_s} \bar{x} + \left.\frac{\partial f}{\partial u}\right|_{x_s, u_s} \bar{u}$$

- Write in state-space form

$$\frac{d\bar{x}}{dt} \approx a\,\bar{x} + b\,\bar{u} \qquad \text{where} \qquad a = \left.\frac{\partial f}{\partial x}\right|_{x_s, u_s} \qquad b = \left.\frac{\partial f}{\partial u}\right|_{x_s, u_s}$$

- If there is a single output that is a function of the state and input

$$y = g(x, u)$$

- Perform a Taylor series expansion and truncate high order terms

$$g(x,u) \approx g(x_s, u_s) + \left.\frac{\partial g}{\partial x}\right|_{x_s, u_s} (x - x_s) + \left.\frac{\partial g}{\partial u}\right|_{x_s, u_s} (u - u_s) \qquad g(x_s, u_s) = y_s$$

$$y - y_s = \left.\frac{\partial g}{\partial x}\right|_{x_s, u_s} (x - x_s) + \left.\frac{\partial g}{\partial u}\right|_{x_s, u_s} (u - u_s)$$

$$\bar{y} = c\,\bar{x} + d\,\bar{u} \qquad \text{where} \qquad c = \left.\frac{\partial g}{\partial x}\right|_{x_s, u_s} \qquad d = \left.\frac{\partial g}{\partial u}\right|_{x_s, u_s}$$

Linearization of Multistate Models

- Two-state system

$$\dot{x}_1 = \frac{dx_1}{dt} = f_1(x_1, x_2, u)$$

$$\dot{x}_2 = \frac{dx_2}{dt} = f_2(x_1, x_2, u)$$

$$y = g(x_1, x_2, u)$$

- Perform Taylor series expansion of the nonlinear functions and neglect high-order terms

$$f_1(x_1, x_2, u) = f_1(x_{1s}, x_{2s}, u_s) + \left.\frac{\partial f_1}{\partial x_1}\right|_{x_{1s}, x_{2s}, u_s} \left(x_1 - x_{1s}\right) + \left.\frac{\partial f_1}{\partial x_2}\right|_{x_{1s}, x_{2s}, u_s} \left(x_2 - x_{2s}\right) + \left.\frac{\partial f_1}{\partial u}\right|_{x_{1s}, x_{2s}, u_s} \left(u - u_s\right)$$

$$f_2(x_1, x_2, u) = f_2(x_{1s}, x_{2s}, u_s) + \left.\frac{\partial f_2}{\partial x_1}\right|_{x_{1s}, x_{2s}, u_s} \left(x_1 - x_{1s}\right) + \left.\frac{\partial f_2}{\partial x_2}\right|_{x_{1s}, x_{2s}, u_s} \left(x_2 - x_{2s}\right) + \left.\frac{\partial f_2}{\partial u}\right|_{x_{1s}, x_{2s}, u_s} \left(u - u_s\right)$$

$$g(x_1, x_2, u) = g(x_{1s}, x_{2s}, u_s) + \left.\frac{\partial g}{\partial x_1}\right|_{x_{1s}, x_{2s}, u_s} \left(x_1 - x_{1s}\right) + \left.\frac{\partial g}{\partial x_2}\right|_{x_{1s}, x_{2s}, u_s} \left(x_2 - x_{2s}\right) + \left.\frac{\partial g}{\partial u}\right|_{x_{1s}, x_{2s}, u_s} \left(u - u_s\right)$$

- For the linearization about the steady-state

$$f_1(x_{1s}, x_{2s}, u_s) = f_2(x_{1s}, x_{2s}, u_s) = 0 \qquad g(x_{1s}, x_{2s}, u_s) = y_s$$

$$\frac{dx_1}{dt} = \frac{d\left(x_1 - x_{1s}\right)}{dt} \qquad \frac{dx_2}{dt} = \frac{d\left(x_2 - x_{2s}\right)}{dt}$$

- We can write the state-space model

$$\begin{bmatrix} \dfrac{d\left(x_1 - x_{1s}\right)}{dt} \\ \dfrac{d\left(x_2 - x_{2s}\right)}{dt} \end{bmatrix} = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial x_1}\right|_{x_{1s}, x_{2s}, u_s} & \left.\dfrac{\partial f_1}{\partial x_2}\right|_{x_{1s}, x_{2s}, u_s} \\ \left.\dfrac{\partial f_2}{\partial x_1}\right|_{x_{1s}, x_{2s}, u_s} & \left.\dfrac{\partial f_2}{\partial x_2}\right|_{x_{1s}, x_{2s}, u_s} \end{bmatrix} \begin{bmatrix} x_1 - x_{1s} \\ x_2 - x_{2s} \end{bmatrix} + \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial u}\right|_{x_{1s}, x_{2s}, u_s} \\ \left.\dfrac{\partial f_2}{\partial u}\right|_{x_{1s}, x_{2s}, u_s} \end{bmatrix} \left[u - u_s\right]$$

$$y - y_s = \begin{bmatrix} \left.\dfrac{\partial g}{\partial x_1}\right|_{x_{1s}, x_{2s}, u_s} & \left.\dfrac{\partial g}{\partial x_2}\right|_{x_{1s}, x_{2s}, u_s} \end{bmatrix} \begin{bmatrix} x_1 - x_{1s} \\ x_2 - x_{2s} \end{bmatrix} + \begin{bmatrix} \left.\dfrac{\partial g}{\partial u}\right|_{x_{1s}, x_{2s}, u_s} \end{bmatrix} \left[u - u_s\right]$$

$$\dot{\bar{\mathbf{x}}} = \mathbf{A}\,\bar{\mathbf{x}} + \mathbf{B}\,\bar{\mathbf{u}}$$

$$\bar{\mathbf{y}} = \mathbf{C}\,\bar{\mathbf{x}} + \mathbf{D}\,\bar{\mathbf{u}}$$

Generalization

- Consider a general nonlinear model with *n* state variables, *m* input variables, and *r* output variables

$$\dot{x}_1 = f_1(x_1, \cdots, x_n, u_1, \cdots, u_m)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, \cdots, x_n, u_1, \cdots, u_m)$$
$$y_1 = g_1(x_1, \cdots, x_n, u_1, \cdots, u_m)$$
$$\vdots$$
$$y_r = g_r(x_1, \cdots, x_n, u_1, \cdots, u_m)$$

Vector notation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u})$$

- Elements of the linearization matrices

$$A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x_s}, \mathbf{u_s}} \qquad B_{ij} = \left.\frac{\partial f_i}{\partial u_j}\right|_{\mathbf{x_s}, \mathbf{u_s}}$$

$$C_{ij} = \left.\frac{\partial g_i}{\partial x_j}\right|_{\mathbf{x_s}, \mathbf{u_s}} \qquad D_{ij} = \left.\frac{\partial g_i}{\partial u_j}\right|_{\mathbf{x_s}, \mathbf{u_s}}$$

State-space form:

$$\dot{\bar{\mathbf{x}}} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{B}\bar{\mathbf{u}}$$
$$\bar{\mathbf{y}} = \mathbf{C}\bar{\mathbf{x}} + \mathbf{D}\bar{\mathbf{u}}$$

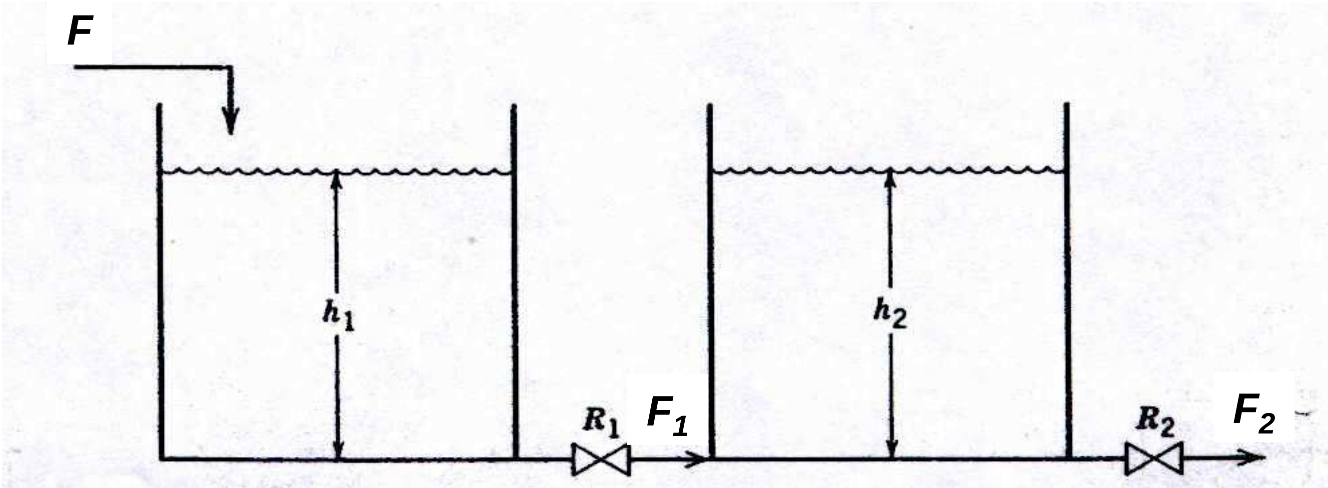
or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

(The “overbar” is usually dropped)

Example: Interacting Tanks

- Two interacting tank in series with outlet flowrate being function of the square root of tank height



$$F_1 = R_1\sqrt{h_1 - h_2}$$

$$F_2 = R_2\sqrt{h_2}$$

- Modeling equations

$$\frac{dh_1}{dt} = \frac{F}{A_1} - \frac{R_1}{A_1}\sqrt{h_1 - h_2} = f_1\left(h_1, h_2, F\right)$$

$$\frac{dh_2}{dt} = \frac{R_1}{A_2}\sqrt{h_1 - h_2} - \frac{R_2}{A_2}\sqrt{h_2} = f_2\left(h_1, h_2, F\right)$$

- Assume only the second tank height is measured. The output, in deviation variable form is $y = h_2 - h_{2s}$

- There are two state variables, one input variable, one one output variable

$$\mathbf{h_s} = \begin{bmatrix} h_{1s} \\ h_{2s} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h_1 - h_{1s} \\ h_2 - h_{2s} \end{bmatrix} \qquad u = F - F_s$$

- The element of the **A** (Jacobian) and **B** matrices

$$A_{11} = \left.\frac{\partial f_1}{\partial h_1}\right|_{\mathbf{h_s}, F_s} = -\frac{R_1}{2A_1\sqrt{h_{1s} - h_{2s}}}$$

$$A_{12} = \left.\frac{\partial f_1}{\partial h_2}\right|_{\mathbf{h_s}, F_s} = \frac{R_1}{2A_1\sqrt{h_{1s} - h_{2s}}}$$

$$A_{21} = \left.\frac{\partial f_2}{\partial h_1}\right|_{\mathbf{h_s}, F_s} = \frac{R_1}{2A_2\sqrt{h_{1s} - h_{2s}}}$$

$$A_{22} = \left.\frac{\partial f_2}{\partial h_2}\right|_{\mathbf{h_s}, F_s} = -\frac{R_1}{2A_2\sqrt{h_{1s} - h_{2s}}} - \frac{R_2}{2A_2\sqrt{h_{2s}}}$$

$$B_{11} = \left.\frac{\partial f_1}{\partial F}\right|_{\mathbf{h_s}, F_s} = \frac{1}{A_1}$$

$$B_{21} = \left.\frac{\partial f_2}{\partial F}\right|_{\mathbf{h_s}, F_s} = 0$$

- Only the height of the second tank is measured

$$y = g(h_1, h_2, F) = h_2 - h_{2s}$$

$$C_{11} = \left.\frac{\partial g}{\partial h_1}\right|_{\mathbf{h_s}, F_s} = 0$$

$$C_{12} = \left.\frac{\partial g_2}{\partial h_2}\right|_{\mathbf{h_s}, F_s} = 1$$

- The state-space model is

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{2A_1\sqrt{h_{1s} - h_{2s}}} & \frac{R_1}{2A_1\sqrt{h_{1s} - h_{2s}}} \\ \frac{R_1}{2A_2\sqrt{h_{1s} - h_{2s}}} & -\frac{R_1}{2A_2\sqrt{h_{1s} - h_{2s}}} - \frac{R_2}{2A_2\sqrt{h_{2s}}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} \\ 0 \end{bmatrix} [u]$$

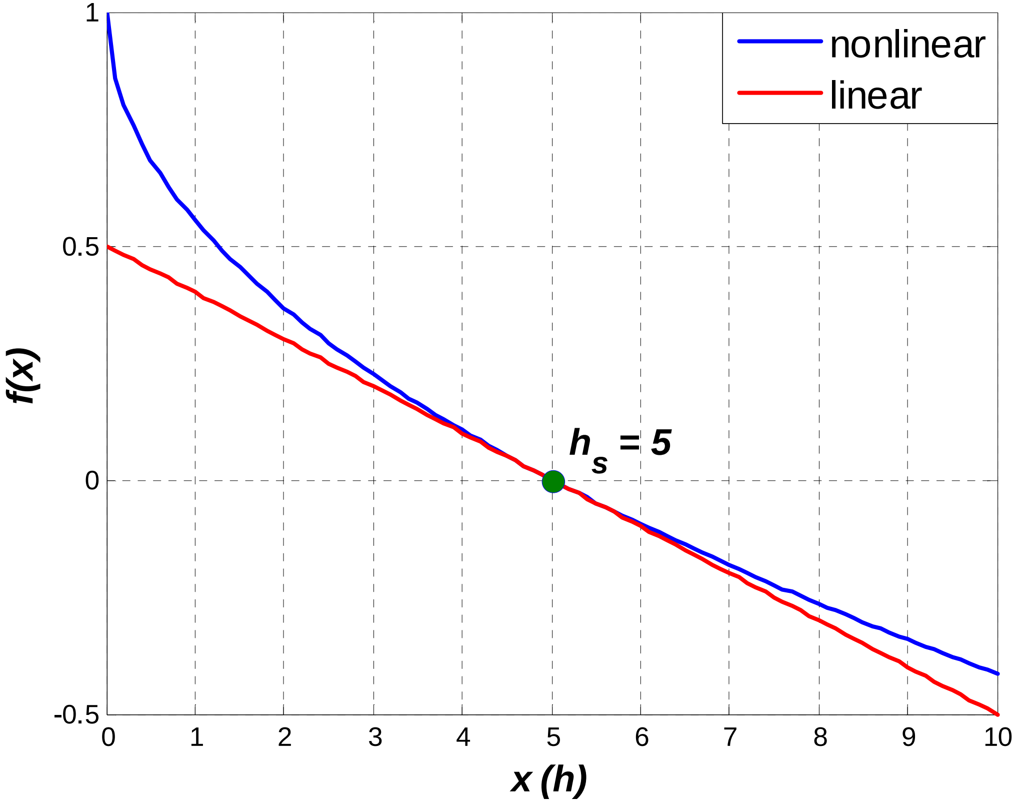
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (y = x_2 = h_2 - h_{2s})$$

Interpretation of Linearization

- Consider the single tank problem (assume *F* is constant)

$$\frac{dh}{dt} = \frac{F}{A} - \frac{R}{A}\sqrt{h} = f\left(h, F\right) = 1 - \frac{1}{\sqrt{5}}\sqrt{h}$$

- Linearization $f\left(h, F\right) \approx 0 - \frac{1}{10}\left(h - h_s\right)$



The linear approximation works well between 3.5 to 7 feet

The two functions are exactly equal at the steady-state value of 5 feet

Exercise: interacting tanks

- Two interacting tank in series with outlet flowrate being function of the square root of tank height
  - Parameter values

$$R_1 = 2.5\frac{\text{ft}^{2.5}}{\text{min}} \quad R_2 = \frac{5}{\sqrt{6}}\frac{\text{ft}^{2.5}}{\text{min}} \quad A_1 = 5\,\text{ft}^2 \quad A_2 = 10\,\text{ft}^2$$

  - Input variable F = 5 ft^3/min
  - Steady-state height values : h_{1s} = 10, h_{2s} = 6

- Perform the following simulation using <u>state-space model</u>
  - What are the responses of tank height if the initial heights are $h_1(0)$=12 ft and $h_2(0)$=7 ft ?
  - Assume the system is at steady-state initially. What are the responses of tank height if
    - F changes from 5 to 7 ft^3/min at t = 0
    - F has periodic oscillation of F = 5 + sin(0.2t)
    - F changes from 5 to 4 ft^3/min at t = 20

Stability of State-Space Models

- A state-space model is said to be stable if the response $x(t)$ is bounded for all $u(t)$ that is bounded
- **Stability criterion for state-space model**
  - The state-space model will exhibit a bounded response $x(t)$ for all bounded $u(t)$, if and only if all of the eigenvalues of **A** have negative real parts

  (the stability is independent matrices **B** and **C**)

- Single variable equation $\dot{x} = a\,x$ has the solution

$$x(t) = e^{at}x(0) \quad \Rightarrow \quad \text{stable if } \; a < 0$$

- The solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$
  - Stable if all of the eigenvalues of **A** are less than zero
  - The response $x(t)$ is oscillatory if the eigenvalues are complex

Exercise

- Consider the following system equations

$$\dot{x}_1 = -0.5x_1 + x_2$$
$$\dot{x}_2 = \qquad\qquad -2x_2$$

  – Find the responses of $\mathbf{x}(t)$ for $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} -0.5547 \\ 0.8321 \end{bmatrix}$

  (slow subspace v.s. fast subspace)

- Consider the following system equations

$$\dot{x}_1 = 2x_1 + x_2$$
$$\dot{x}_2 = 2x_1 - x_2$$

  – Find the responses of $\mathbf{x}(t)$ for $\mathbf{x}(0) = \begin{bmatrix} 0.2703 \\ -0.9628 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 0.8719 \\ 0.4896 \end{bmatrix}$

  (stable subspace v.s. unstable subspace)

Note: Find eigenvalue and eigenvector of A
>> [V, D] = eig(A)